Propagating interface in a monostable reaction-diffusion equation with time delay

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Abstract

We consider a monostable time-delayed reaction-diffusion equation arising from population dynamics models. We let a small parameter tend to zero and investigate the behavior of the solutions. We construct accurate lower barriers — by using a non standard bistable approximation of the monostable problem— and upper barriers. As a consequence, we prove the convergence to a propagating interface.

Key Words: time-delayed reaction-diffusion equation, delay differential equation, travelling wave, propagating interface.[1]

1 Introduction

In this work we investigate the singular limit, as $\varepsilon \to 0^+$, of $u^\varepsilon : [-\varepsilon\tau, \infty)\times\mathbb{R}^N \to \mathbb{R}$ the solution of the delayed reaction-diffusion equation

$$\partial_t u(t,x) = \varepsilon\Delta u(t,x) + \frac{1}{\varepsilon}\left[f(u(t-\varepsilon\tau, x)) - u(t,x)\right], \quad t>0,\ x\in\mathbb{R}^N, \tag{1.1}$$

supplemented with the initial data of delayed type

$$u(\theta, x) = \varphi\left(\frac{\theta}{\varepsilon}, x\right), \quad -\varepsilon\tau \le \theta \le 0,\ x\in\mathbb{R}^N. \tag{1.2}$$

Here $\tau > 0$ is a given delay parameter; $f : [0,\infty) \to [0,\infty)$ is a given increasing and monostable nonlinearity — see (1.4) for precise assumptions; the initial data $\varphi : [-\tau, 0]\times\mathbb{R}^N \to \mathbb{R}$ is a given smooth function — see Assumption 1.1.

Equation (1.1) is widely used in population dynamics models. In this context, $u(t,x)$ denotes the density of individuals at time t and spatial location x.

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The function f is the birth rate of the population. Note that the birth feedback appears with some time delay in order to take into account the period of maturation to become adult. Finally the term $-u$ corresponds to a normalized death rate, while $\varepsilon > 0$ is a scaling parameter.

When f takes the form of the so-called Ricker's function

$$f(u) = \widehat{\alpha} u e^{-u}, \quad \widehat{\alpha} > 1, \tag{1.3}$$

equation (1.1) is commonly referred as the *diffusive Nicholson's blowflies equation.* This kind of equation has been intensively studied in the literature. The purely reactive part, namely the underlying delay differential equation, has attracted the attention of many researchers during the past decades (see for instance [15] and references cited therein). On the other hand, the diffusive equation has also been extensively investigated from the spatial propagation point of view, that is speed of spread, travelling wave solutions (we refer for instance to So and Zou [20], So, Wu and Zou [19], Thieme and Zhao [22], Fang and Zhao [7], and the references therein).

In this work, we consider the monostable equation (1.1) in the so-called monotonic regime. Precisely we assume that $f : [0, \infty) \to [0, \infty)$ is a function of the class C^2 such that

$$\begin{cases} f(0) = 0, \quad f(1) = 1, \quad f'(0) > 1, \quad f'(1) < 1, \\ f'(u) > 0, \quad \forall u \in (0, 1), \\ f(u) > u, \quad \forall u \in (0, 1). \end{cases} \tag{1.4}$$

In particular, $u \equiv 0$ and $u \equiv 1$ solve (1.1). If we come back to example (1.3), assuming $\widehat{\alpha} \in (1, e)$ implies that f satisfies (1.4), with $\ln \widehat{\alpha}$ playing the role of 1.

Let us observe that, when $\tau = 0$, equation (1.1) reduces to the monostable reaction-diffusion equation

$$\partial_t u(t, x) = \varepsilon \Delta u(t, x) + \frac{1}{\varepsilon} F\left(u(t, x)\right), \tag{1.5}$$

with $F(u) := f(u) - u$. In view of (1.4), the nonlinearity F exhibits a monostable dynamics, namely $F(0) = F(1) = 0$, $F(u) > 0$ for all $u \in (0, 1)$, and $F'(0) > 0$ while $F'(1) < 0$. Under these assumptions, solutions of (1.5) with compactly supported initial data have been considered first by Freidlin [9] with probabilistic tools, then by Evans and Souganidis [6] with Hamilton-Jacobi techniques (we also refer to [4, 5] and the references therein). This problem has been recently revisited using comparison parabolic arguments in [2] (including the case of compactly supported initial data), and [1] (for slowly decaying initial data). Roughly speaking, for compactly supported initial data with convex and bounded support, as $\varepsilon \to 0$, the solution of (1.5) generates a sharp interface at the very early stages of the dynamics. Then the interface propagates through the spatial domain, according to a free boundary problem with constant speed in the normal direction. This speed turns out to be the minimal speed of propagation of some underlying travelling wave solutions.

In the delayed case ($\tau > 0$) that we consider, we will show that the above scenario remains valid under the following assumption on the initial data φ arising in (1.2).

Assumption 1.1. *We assume that* $\varphi : [-\tau, 0] \times \mathbb{R}^N \to [0, 1]$ *is a uniformly continuous function satisfying the following.*

(i) There exists $w_0 \in BUC^2(\mathbb{R}^N, \mathbb{R})$ such that

$$\Omega_0 := \{x \in \mathbb{R}^N : \ w_0(x) > 0\}$$

is a nonempty smooth bounded and convex domain, and

$$w_0(x) \leq \varphi(\theta, x), \ \ \forall(\theta, x) \in [-\tau, 0] \times \mathbb{R}^N. \tag{1.6}$$

(ii) There exists $\delta > 0$ such that

$$|\nabla w_0(x).\nu_{\partial\Omega_0}(x)| \geq \delta, \ \ \forall x \in \Gamma_0 := \partial\Omega_0, \tag{1.7}$$

wherein $\nu_{\partial\Omega_0}(x)$ denotes the outward unit normal vector to Ω_0 at $x \in \Gamma_0$.

(iii) There exists $v_0 \in BUC(\mathbb{R}^N, [0, 1))$ such that

$$\text{supp } v_0 = \overline{\Omega_0}\,, \tag{1.8}$$

and

$$\varphi(\theta, x) \leq v_0(x), \ \ \forall(\theta, x) \in [-\tau, 0] \times \mathbb{R}^N. \tag{1.9}$$

Remark 1.2. The hypothesis $\|v_0\|_\infty < 1$ in *(iii)* shall be used in the construction of upper barriers in Section 5. Nevertheless, when $\|v_0\|_\infty = 1$, our main result remains valid under the additional assumption that f satisfies

$$f(K_0 u) \leq K_0 f(u), \ \ \forall u \in [0, 1], \tag{1.10}$$

for some constant $K_0 > 1$. See Remark 5.2 for details.

Before stating our main convergence result let us give some notations. Under assumption (1.4), we denote by $c^* > 0$ the minimal speed of the underlying delayed travelling waves (see Lemma 2.3 for details). In particular, there is $(U^*, c^*) \in C^2(\mathbb{R}) \times (0, \infty)$ such that U^* is nonincreasing and

$$\begin{cases} (U^*)''(z) + c^*(U^*)'(z) + f\left(U^*(z + c^*\tau)\right) - U^*(z) = 0, \ \ \forall z \in \mathbb{R}, \\ U^*(-\infty) = 1 \text{ and } U^*(\infty) = 0. \end{cases}$$

Next, for $c > 0$, we denote by $\Gamma^c := \bigcup_{t \geq 0}(\{t\} \times \Gamma_t^c)$ the smooth solution of the free boundary problem (see subsection 4.1 for details)

$$(P^c) \quad \begin{cases} V = c \quad \text{ on } \Gamma_t^c \\ \Gamma_t^c\big|_{t=0} = \Gamma_0, \end{cases}$$

with V the normal velocity of Γ_t^c in the exterior direction, and Γ_0 the initial interface defined in (1.7). Also, we denote by Ω_t^c the region enclosed by the hypersurface Γ_t^c.

Here is the main result of the present paper (see subsection 2.1 for the well-posedness of the initial value problem (1.1)–(1.2)).

Theorem 1.3 (Convergence to a propagating interface)**.** *Let the nonlinearity f be as in* (1.4)*. Let the initial data φ satisfy Assumption 1.1. For each $\varepsilon > 0$, let $u^\varepsilon : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$ be the solution of* (1.1)*–*(1.2)*. Then the following convergence results hold.*

(i) For each $c \in (0, c^*)$ and each $t_0 > 0$, we have

$$\lim_{\varepsilon \to 0^+} \sup_{t \geq t_0} \sup_{x \in \overline{\Omega_t^c}} |1 - u^\varepsilon(t, x)| = 0.$$

(ii) For each $c > c^*$ and each $t_0 > 0$, we have

$$\lim_{\varepsilon \to 0^+} \sup_{t \geq t_0} \sup_{x \in \mathbb{R}^N \setminus \Omega_t^c} u^\varepsilon(t, x) = 0.$$

A first step towards Theorem 1.3 consists in proving that, as $\varepsilon \to 0$, the initial value problem (1.1)–(1.2) generates a sharp interface after a very small time of order $O(\varepsilon |\ln \varepsilon|)$. Then, to analyze the propagation of the interface, we aim at constructing suitable sub- and super-solutions. This step is strongly related to the existence of travelling wave solutions. While the upper barriers are directly constructed by using monostable travelling fronts, the construction of lower barriers is much more delicate. This kind of problem has been solved in several situations. In [11], the authors consider a *degenerate* reaction-diffusion equation, and take advantage of the existence of *sharp* travelling fronts to construct sub-solutions. In [2], the standard Fisher-KPP case is considered. The construction of lower barriers of propagation is performed by using the existence of non-monotone (and also not everywhere positive) travelling waves with speeds $c < c^*$. In the non delayed case, the existence of such a connection easily follows from a phase plane analysis. In the delayed case we consider, the existence of similar waves is far from obvious. The key idea of the present paper is to construct sub-solutions of propagation by using travelling waves for a modified time delayed reaction-diffusion equation with a bistable dynamics. We hope that such a strategy could be used to understand better the classical non delayed Fisher-KPP case and also to analyze a larger class of equations.

The organization of the present paper is as follows. In Section 2, we recall known facts on the well-posedness of the initial value problem (1.1)–(1.2). We also discuss the links between monostable travelling waves associated with f, and bistable ones associated with approximations f_η of f. This is necessary to develop the key strategy mentioned above. In Section 3, we investigate the generation of a sharp interface in the very early stages of the dynamics. This is strongly related with the underlying delay differential equation. Section 4 is concerned with the study of the propagation of interface from below. We shall construct accurate lower barriers by using a bistable approximation. As a result of Sections 3 and 4, we shall prove Theorem 1.3 (i). Section 5 deals with the construction of upper barriers to control the propagation from above. This will imply Theorem 1.3 (ii).

2 Preliminary

2.1 Existence and comparison for (1.1)–(1.2)

We first state the following comparison principle for monotone delayed reaction-diffusion equations.

Proposition 2.1 (Comparison principle). *Let $\tau > 0$, $T > 0$ and $g : \mathbb{R} \to \mathbb{R}$ an increasing and continuous function be given. Let $(u, v) \in C\left([-\tau, T] \times \mathbb{R}^N\right)$ be two bounded functions satisfying*

$$\partial_t u,\ \partial_t v,\ \nabla u,\ \nabla v,\ D^2 u,\ D^2 v\ \in L^2_{loc}\left((0,T) \times \mathbb{R}^N\right).$$

Assume

$$\begin{aligned} (\partial_t - \Delta + 1)\, u(t,x) - g\left(u(t-\tau, x)\right) \le 0 \\ (\partial_t - \Delta + 1)\, v(t,x) - g\left(v(t-\tau, x)\right) \ge 0, \end{aligned} \tag{2.1}$$

for almost every $(t, x) \in (0,T) \times \mathbb{R}^N$, and

$$u(\theta, x) \le v(\theta, x) \quad \text{for all } (\theta, x) \in [-\tau, 0] \times \mathbb{R}^N. \tag{2.2}$$

Then $u(t,x) \le v(t,x)$, for all $(t,x) \in [-\tau, T] \times \mathbb{R}^N$.

Proof. Let us consider the map $w := u - v \in C\left([-\tau, T] \times \mathbb{R}^N\right)$. Since g is increasing, it follows from (2.1) and (2.2) that w satisfies

$$(\partial_t - \Delta + 1)\, w(t,x) \le 0 \quad a.e. \text{ in } (0, \min(T, \tau)) \times \mathbb{R}^N.$$

Since $w(0, \cdot) \le 0$, the weak comparison principle [14, Proposition 52.8] implies $w \le 0$ in $(0, \min(T,\tau)) \times \mathbb{R}^N$. If $T > \tau$, one can repeat the argument on $(\tau, \min(T, 2\tau)) \times \mathbb{R}^N$. This proves the proposition. □

We now introduce some notations. Let $X := \mathrm{BUC}\left(\mathbb{R}^N, \mathbb{R}\right)$ be the Banach space of bounded and uniformly continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$, endowed with the usual supremum norm. Define also the Banach spaces $\mathcal{C} := C([-\tau, 0], X)$ and $\mathcal{C}_0 := C([-\tau, 0], \mathbb{R})$. For convenience, we identify $\psi \in \mathcal{C}$ as a function from $[-\tau, 0] \times \mathbb{R}^N$ into $\mathbb{R}$ defined by $\psi(\theta, x) = \psi(\theta)(x)$. For each $\alpha < \beta$, we define

$$[\alpha, \beta]_{\mathcal{C}} := \left\{\psi \in \mathcal{C} :\ \ \alpha \le \psi(\theta, x) \le \beta,\ \forall (\theta, x) \in [-\tau, 0] \times \mathbb{R}^N\right\},$$

and $[\alpha, \beta]_{\mathcal{C}_0} := \mathcal{C}_0 \cap [\alpha, \beta]_{\mathcal{C}}$. Next, for any continuous function $w : [-\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$, we define $w_t \in \mathcal{C}$, $t \ge 0$, by

$$w_t : (\theta, x) \in [-\tau, 0] \times \mathbb{R}^N \mapsto w_t(\theta, x) = w(t + \theta, x).$$

The well-posedness of the initial value problem (1.1)–(1.2) can classically be investigated via the theory of abstract functional differential equations: since the initial data $\varphi \in [0,1]_{\mathcal{C}}$, the initial value problem (1.1)–(1.2) admits a unique *mild* solution $u^\varepsilon : [0, \infty) \times \mathbb{R}^N \to [0,1]$, which is actually classical on $[\varepsilon\tau, \infty) \times \mathbb{R}^N$. For more details, we refer the reader to the monograph of Wu [23] and the references cited therein.

2.2 Monostable and bistable delayed travelling waves

As explained in the introduction, the construction of lower barriers is far from obvious when $\tau > 0$. A key idea of the present paper is to derive the *monostable propagation of the interface from below* from the bistable case. To perform this in Section 4, let us first define a family of bistable approximations by extending the monostable nonlinearity f for negative values of u.

Bistable approximations of f. For $\eta \in (0,1]$, we introduce an increasing and bounded map $f_\eta : \mathbb{R} \to \mathbb{R}$ of the class C^2 such that

$$\begin{aligned} & f_\eta(u) = f(u) \;\; \forall u \in [0,1] \\ & f_\eta(-\eta) = -\eta \quad \text{and } {f_\eta}'(-\eta) < 1 \\ & f_\eta(u) < u \;\; \forall u \in (-\eta, 0) \cup (1, \infty) \quad \text{and } f_\eta(u) > u \;\; \forall u \in (-\infty, -\eta) \cup (0,1). \end{aligned} \tag{2.3}$$

Observe that f_η has exactly three fixed points $-\eta < 0 < 1$, ${f_\eta}'(-\eta) < 1$ and ${f_\eta}'(1) = f'(1) < 1$. We also require that the family $\{f_\eta\}_{\eta \in (0,1]}$ is ordered in the sense that:

$$\forall\, (\eta, \eta') \in (0,1]^2, \;\; \eta < \eta' \;\Rightarrow\; f_{\eta'}(u) \le f_\eta(u) \;\; \forall u \in \mathbb{R}. \tag{2.4}$$

Travelling waves. We consider the one dimensional reaction-diffusion equation with time delay

$$(\partial_t - \partial_{xx} + 1)\, u(t,x) = f_\eta\left(u(t-\tau, x)\right), \; t > 0, \; x \in \mathbb{R}. \tag{2.5}$$

We denote by $u_\eta \equiv u_\eta(t,x;\psi) : [-\tau, \infty) \times \mathbb{R}^N \to [-\eta, 1]$ the solution of (2.5) with the initial condition

$$u_0(\theta)(x) = u(\theta, x) = \psi \in [-\eta, 1]_{\mathcal{C}}. \tag{2.6}$$

Let us notice that the above initial value problem generates a strongly continuous and increasing semiflow $\{Q_\eta(t)\}_{t \ge 0}$ defined by

$$\left[Q_\eta(t)\psi\right](\theta, x) = \left(u_\eta\right)_t(\theta, x; \psi), \; (\theta, x) \in [-\tau, 0] \times \mathbb{R}^N,$$

and acting $[-\eta, 1]_{\mathcal{C}}$ into itself. Also, it follows from (2.3) that, for each $t \ge 0$, $Q_\eta(t)[0,1]_{\mathcal{C}} \subset [0,1]_{\mathcal{C}}$ and that $Q(t) := \left(Q_\eta(t)\right)|_{[0,1]_{\mathcal{C}}}$ does not depend upon η. Note that Q_η exhibits a bistable dynamics while Q is of monostable type.

Let us state some basic facts on travelling waves sustained by (2.5).

Lemma 2.2 (Bistable Travelling waves)**.** *For $\eta \in (0,1]$ arbitrary, the following holds.*

(i) There exists a unique speed c_η such that (2.5) has a travelling wave solution $(U_\eta, c_\eta) \in C^2(\mathbb{R}) \times \mathbb{R}$ whose profile U_η is nonincreasing, that is

$$\begin{cases} {U_\eta}''(z) + c_\eta {U_\eta}'(z) + f_\eta\left(U_\eta(z + c_\eta \tau)\right) - U_\eta(z) = 0, \;\; \forall z \in \mathbb{R}, \\ U_\eta(-\infty) = 1 \;\; \textit{and} \;\; U_\eta(\infty) = -\eta. \end{cases} \tag{2.7}$$

(ii) There exist two constants $(\mu, M) \in (0, \infty)^2$ such that

$$\begin{cases} |1 - U_\eta(z)| + |-\eta - U_\eta(-z)| \le M e^{\mu z}, \; \forall z \le 0, \\ |{U_\eta}'(z)| + |{U_\eta}''(z)| \le M e^{-\mu |z|}, \;\; \forall z \in \mathbb{R}. \end{cases}$$

(iii) There exists some constant $\gamma > 0$ such that, for any $\psi \in [-\eta, 1]_{\mathcal{C}}$ with

$$\liminf_{x \to -\infty} \min_{\theta \in [-\tau, 0]} \psi(\theta, x) > 0 \;\; \textit{and} \;\; \limsup_{x \to +\infty} \max_{\theta \in [-\tau, 0]} \psi(\theta, x) < 0, \tag{2.8}$$

one can find $K = K(\psi) > 0$ and $\xi = \xi(\psi) \in \mathbb{R}$ such that

$$|u_\eta(t, x; \psi) - U_\eta(x - c_\eta t + \xi)| \le K e^{-\gamma t}, \;\; \forall (t, x) \in [0, \infty) \times \mathbb{R}.$$

Proof. Part (i) comes from Schaaf [16, Theorem 3.13] (see also Fang and Zhao [8, Theorem 6.4]). The behavior of the profile (ii) can be found in Hupkes and Lunel [12, Proposition 2.2.5]. Finally the *global asymptotic stability with phase shift of the wave* (iii) is due to Smith and Zhao [18, Theorem 3.3]. $\square$

We recall that f satisfies (1.4). As far as monostable travelling waves sustained by

$$(\partial_t - \partial_{xx} + 1)\, u(t,x) = f\left(u(t-\tau,x)\right),\ t>0,\ x\in\mathbb{R}, \tag{2.9}$$

are concerned, we quote the following result from Schaaf [16, Theorem 2.7] (see also [13]).

Lemma 2.3 (Monostable travelling waves)**.** *There exists $c^*>0$ such that* (2.9) *has a travelling wave solution $(U_c,c)\in C^2(\mathbb{R})\times(0,\infty)$ with $0\le U_c\le 1$, if and only if $c\ge c^*$. In addition, when $c\ge c^*$ the waves are nonincreasing.*

In the sequel we denote by (U^*,c^*) the monostable wave with minimal speed, that is

$$\begin{cases}(U^*)''(z)+c^*(U^*)'(z)+f\left(U^*(z+c^*\tau)\right)-U^*(z)=0, \quad \forall z\in\mathbb{R},\\ U^*(-\infty)=1 \ \text{ and } \ U^*(\infty)=0.\end{cases} \tag{2.10}$$

To conclude this preliminary, we prove the following result on the convergence of the bistable speeds c_η.

Lemma 2.4 (Convergence of speeds)**.** *Let f satisfy* (1.4)*. Let $\{f_\eta\}_{\eta\in(0,1]}$ satisfy* (2.3) *and* (2.4)*. Then the family $\{c_\eta\}_{\eta\in(0,1]}$ is decreasing and*

$$c_\eta \nearrow c^*, \quad \textit{as } \eta\searrow 0.$$

Proof. Let $\eta\in(0,1]$ be given. Since $0\le U^*\le 1$ and $f_\eta|_{[0,1]}=f$, $U^*(x-c^*t)$ solves (2.5). We can select a $\psi\in[-\eta,1]_{\mathcal{C}}$ such that (2.8) holds together with

$$\psi(\theta,x)\le U^*(x-c^*\theta),\ \forall(\theta,x)\in[-\tau,0]\times\mathbb{R}^N.$$

The comparison principle yields $u_\eta(t,x;\psi)\le U^*(x-c^*t)$, so that Lemma 2.2 (iii) implies

$$U_\eta(x-c_\eta t+\xi)-Ke^{-\gamma t}\le U^*(x-c^*t),$$

for some constants $\gamma>0$, $K>0$ and $\xi\in\mathbb{R}$. Choosing $x=c^*t$, we get $U_\eta((c^*-c_\eta)t+\xi)-Ke^{-\gamma t}\le U^*(0)$; if $c^*<c_\eta$ then letting $t\to\infty$, we collect $1\le U^*(0)$, a contradiction. Hence, we have $c_\eta\le c^*$.

Now, let us take $\eta<\eta'$ in $(0,1]$. In view of (2.4), the comparison principle implies $u_{\eta'}(t,x;\psi)\le u_\eta(t,x;\psi)$ for any $\psi\in[-\eta,1]_{\mathcal{C}}$. Choosing ψ given by $\psi(\theta,x)=U_\eta(x-c_\eta\theta)$ and using Lemma 2.2 (iii), we infer that

$$U_{\eta'}(x-c_{\eta'}t+\xi')-K'e^{-\gamma' t}\le U_\eta(x-c_\eta t),$$

for some given constants $\gamma'>0$, $K'>0$ and $\xi\in\mathbb{R}$. Choosing $h\in\mathbb{R}$ such that $U_{\eta'}(h)=0$, $x=c_{\eta'}t-\xi'+h$, we get $-K'e^{-\gamma' t}\le U_\eta((c_{\eta'}-c_\eta)t-\xi'+h))$; if $c_{\eta'}>c_\eta$ then letting $t\to\infty$, we collect $0\le-\eta$, a contradiction. Hence, we have $c_{\eta'}\le c_\eta$.

As a result, there is $\hat{c} \leq c^*$ such that $c_\eta \nearrow \hat{c}$, as $\eta \searrow 0$. To conclude let us make the normalization $U_\eta(0) = 1/2$ for each η. Classically, by the interior elliptic estimates and Sobolev embedding theorem, we may assume that, modulo extraction, $U_\eta \to \hat{U}$ strongly in $C^{1,\beta}_{loc}(\mathbb{R})$ and weakly in $W^{2,p}_{loc}(\mathbb{R})$, $1 < p < \infty$. Then $(\hat{U}, \hat{c})$ satisfies (2.10) with c^* replaced by $\hat{c}$. Lemma 2.3 then enforces $\hat{c} \geq c^*$. The lemma is proved. □

3 Lower barriers for small times

The goal of this section is to prove that, after a very short time as $\varepsilon \to 0$, the solution $u^\varepsilon : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to [0, 1]$ of (1.1)–(1.2) is very close to 1 in Ω_0 (roughly speaking). Precisely, the following holds.

Proposition 3.1 (Generation of interface from below)**.** *Let the initial data φ satisfy Assumption 1.1 (i) – (ii). Denote by $d(0, x)$ the smooth cut-off signed distance function to Γ_0 as defined in subsection 4.1 (in particular, $d(0, x) < 0$ if and only if $x \in \Omega_0$). Then there exist $\delta_0 > 0$, $\alpha_0 > 0$, $\rho_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $(\theta, x) \in [-\tau, 0] \times \mathbb{R}^N$, the following holds.*

$$\textit{If } d(0, x) \leq -\delta_0\varepsilon|\ln\varepsilon| \textit{ then } 1 - \varepsilon^{\rho_0} \leq u^\varepsilon(\alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau + \varepsilon\theta, x) \leq 1.$$

The proof shall be given in the end of the present section. The idea is to construct a sub-solution based upon the delay differential equation obtained by neglecting diffusion in (1.1).

3.1 A delay differential equation

Let us consider the delay differential equation

$$\begin{cases} \dfrac{dv}{dt}(t) = f\left(v_t(-\tau)\right) - v(t), \quad t > 0, \\ v_0(\cdot) = \phi(\cdot) \in [0, 1]_{\mathcal{C}_0}, \end{cases} \tag{3.1}$$

where f satisfies (1.4) (recall that $\mathcal{C}_0 = C([-\tau, 0], \mathbb{R})$). Because of the aforementioned reason, we also need to consider, for $\eta \in (0, 1]$, the delay differential equation

$$\begin{cases} \dfrac{dv}{dt}(t) = f_\eta\left(v_t(-\tau)\right) - v(t), \quad t > 0, \\ v_0(\cdot) = \phi(\cdot) \in [-\eta, 1]_{\mathcal{C}_0}, \end{cases} \tag{3.2}$$

where f_η was defined in (2.3). From standard results for delay differential equation with quasi-monotone nonlinearity — see for instance the monograph of Smith [17]— the following holds.

Lemma 3.2 (Well-posedness)**.** *For each $\phi \in \mathcal{C}_0$, (3.2) has a unique global (mild) solution $v_\eta = v_\eta(\cdot\,; \phi) : [-\tau, \infty) \to \mathbb{R}$ and the semiflow $V_\eta(t)\phi = V_\eta(t; \phi) := (v_\eta)_t(\cdot\,; \phi)$ is strongly continuous and monotone increasing on $\mathcal{C}_0$. It furthermore satisfies the following properties.*

(i) For each $t \geq 0$, $V_\eta(t)[-\eta, 1]_{\mathcal{C}_0} \subset [-\eta, 1]_{\mathcal{C}_0}$.

(ii) For each $t \geq 0$, $V_\eta(t)[0,1]_{\mathcal{C}_0} \subset [0,1]_{\mathcal{C}_0}$. The restriction $V(t) := V_\eta(t)|_{[0,1]_{\mathcal{C}_0}}$ does not depend upon η and, for $\phi \in [0,1]_{\mathcal{C}_0}$, the map $t \mapsto V(t)\phi = V(t;\phi)$ is the mild solution $v_t(\cdot\,;\phi)$ of (3.1).

Dynamics of the DDE. We start with a lemma on the global dynamics of (3.1) on $[0,1]_{\mathcal{C}_0}$.

Lemma 3.3 (Stability of 1)**.** *The following holds.*

(i) For $\phi \in [0,1]_{\mathcal{C}_0} \setminus \{0\}$, we have $\lim_{t\to\infty} V(t)\phi = 1$ in $\mathcal{C}_0$.

(ii) There exist $\delta_1 > 0$, $M > 0$ and $\lambda > 0$ such that, for all $\phi \in \mathcal{C}_0$,

$$\|1-\phi\|_{L^\infty(-\tau,0)} \leq \delta_1 \quad \Rightarrow \quad \|1 - V(t)\phi\|_{L^\infty(-\tau,0)} \leq Me^{-\lambda t}, \;\; \forall t \geq 0.$$

Proof. Let us prove (i), that is the global stability of the stationary point $\overline{v} = 1$. First, we consider the case where there is $\zeta \in (0,1)$ such that $\phi(\theta) \geq \zeta$, for all $\theta \in [-\tau, 0]$. Since the semiflow associated with (3.1) is monotone increasing and since $V(t)[0,1]_{\mathcal{C}_0} \subset [0,1]_{\mathcal{C}_0}$, it is enough to consider the solution with the constant ζ as initial data, that is $V(t;\zeta) = v_t(\cdot\,;\zeta)$. Since $f(\zeta) > \zeta$, the map $t \mapsto v(t;\zeta)$ is nondecreasing. Hence we get $\lim_{t\to\infty} v(t;\zeta) = 1$, which in turn implies $\|V(t)\zeta - 1\|_\infty = \sup_{-\tau\leq\theta\leq 0} |v(t+\theta,\zeta) - 1| \to 0$, as $t \to \infty$. This concludes the proof of (i) for this first case. Let us now consider the general case. Since $\phi \in [0,1]_{\mathcal{C}_0} \setminus \{0\}$, there exist $-\tau < a < b < 0$ and $\beta > 0$ such that

$$\phi(\theta) \geq \beta \mathbf{1}_{[a,b]}(\theta), \;\; \forall \theta \in [-\tau, 0].$$

From (3.1), we obtain that, for all $t \in (0, \tau]$,

$$\frac{d}{dt}\left(e^t v(t;\phi)\right) \geq e^t f(\beta)\mathbf{1}_{[\tau+a,\tau+b]}(t) \geq f(\beta)\mathbf{1}_{[\tau+a,\tau+b]}(t).$$

Integrating this from 0 to τ yields $v(\tau;\phi) \geq f(\beta)(b-a)$. Now, for all $t \in (\tau, 2\tau]$, (3.1) implies $\frac{d}{dt}\left(e^t v(t;\phi)\right) \geq 0$. Hence

$$v(t;\phi) \geq e^{\tau - t} v(\tau;\phi) \geq \zeta := e^{-\tau} f(\beta)(b-a) > 0, \;\; \forall t \in [\tau, 2\tau],$$

and we are back to the first case. This completes the proof of (i).

The proof of (ii) is a direct consequence of the exponential local stability of $\overline{v}$. Indeed, at this point the characteristic equation associated to (3.1) reads as

$$\Delta(\lambda) := \lambda + 1 - f'(1)e^{-\lambda\tau} = 0.$$

Since $f'(1) < 1$, all roots have strictly negative real parts and the result follows (see for instance [21], [10] and the references therein). □

Next, we shall prove the following important result.

Proposition 3.4 (Convergence to 1)**.** *Let $\phi \geq 0$ in $\mathcal{C}_0 \setminus \{0\}$ be given. There exists $\lambda > 0$ such that, for all $\alpha > 0$ there exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$1 - \varepsilon^{\alpha\lambda/2} \leq V\left(\alpha|\ln\varepsilon| + t; \varepsilon|\ln\varepsilon|\phi\right)(\theta) \leq 1, \;\; \forall (\theta, t) \in [-\tau, 0] \times [0, \infty).$$

Proof. Let $\phi \geq 0$ in $\mathcal{C}_0 \setminus \{0\}$ be given. Recalling that $f'(0) > 1$, let $\delta \in (0,1)$ and $\rho > 1$ be such that

$$f(u) \geq \rho u, \quad \forall u \in [0, \delta]. \tag{3.3}$$

Applying Lemma 3.3 with δ as initial data, we have the existence of constants $M > 0$ and $\lambda > 0$ such that

$$0 \leq 1 - V(t;\delta)(\theta) \leq Me^{-\lambda t}, \quad \forall (\theta, t) \in [-\tau, 0] \times [0, \infty). \tag{3.4}$$

Let $\alpha > 0$ be given. Consider $\varepsilon_0 > 0$ small enough so that $\varepsilon|\ln \varepsilon|\phi \in [0, \delta]_{\mathcal{C}_0}$ for all $\varepsilon \in (0, \varepsilon_0)$. Since $\phi \geq 0$ is in $\mathcal{C}_0 \setminus \{0\}$, there exist $-\tau < a < b < 0$ and $\beta > 0$ such that

$$\varepsilon|\ln \varepsilon|\phi(\theta) \geq \varepsilon|\ln \varepsilon|\beta \mathbf{1}_{[a,b]}(\theta), \quad \forall \theta \in [-\tau, 0].$$

Arguing as in the proof of Lemma 3.3 and using (3.3), we discover that there is $\zeta > 0$ such that, for $\varepsilon > 0$ small enough,

$$v_\varepsilon(t) := v(t; \varepsilon|\ln \varepsilon|\phi) \geq \zeta \varepsilon |\ln \varepsilon|, \quad \forall t \in [\tau, 2\tau]. \tag{3.5}$$

Next, observe that, for all $0 < t \leq \tau$,

$$\frac{d}{dt}\left(e^t v_\varepsilon(t)\right) = e^t f(\varepsilon|\ln \varepsilon|\phi(t-\tau)) \leq e^\tau \varepsilon|\ln \varepsilon| \|\phi\|_\infty \|f'\|_\infty =: C\varepsilon|\ln \varepsilon|.$$

Integrating this from 0 to τ, we have $v_\varepsilon(\tau) \leq e^{-\tau}(\phi(0) + C\tau)\varepsilon|\ln \varepsilon| < \delta$, for $\varepsilon > 0$ small enough. Therefore we can define

$$t^\varepsilon := \sup \{t > 2\tau : \ v_\varepsilon(s - \tau) \leq \delta, \ \forall s \in [2\tau, t]\}.$$

It then follows from the DDE (3.1) and (3.3) that

$$v_\varepsilon{}'(t) \geq \rho v_\varepsilon(t - \tau) - v_\varepsilon(t), \quad \forall t \in [2\tau, t^\varepsilon]. \tag{3.6}$$

Since $\rho > 1$, there is $a > 0$ such that $a + 1 = \rho e^{-a\tau}$. Then the map $h : t \mapsto A\varepsilon|\ln \varepsilon|e^{at}$, $A := \zeta/e^{2a\tau}$ satisfies

$$h'(t) = \rho h(t - \tau) - h(t), \quad \forall t \in [2\tau, t^\varepsilon], \text{ and } h(t) \leq \zeta\varepsilon|\ln \varepsilon|, \quad t \in [\tau, 2\tau]. \tag{3.7}$$

It follows from (3.6), (3.5) and (3.7) that

$$v_\varepsilon(t) \geq A\varepsilon|\ln \varepsilon|e^{at}, \quad \forall t \in [2\tau, t^\varepsilon].$$

In view of $v_\varepsilon(t^\varepsilon - \tau) = \delta$, we have

$$t^\varepsilon \leq \tau + \frac{1}{a} \ln \frac{\delta}{A\varepsilon|\ln \varepsilon|}. \tag{3.8}$$

Now since the map $t \mapsto v_\varepsilon(t)$ is increasing, we deduce from $v_\varepsilon(t^\varepsilon - \tau) = \delta$ that

$$v_\varepsilon(t^\varepsilon + t + \theta) \geq \delta, \quad \forall (\theta, t) \in [-\tau, 0] \times [0, \infty).$$

In view of (3.8), we have $t^\varepsilon \leq \alpha|\ln \varepsilon|$ for $\varepsilon > 0$ small enough so that

$$v_\varepsilon(\alpha|\ln \varepsilon| + t + \theta) \geq \delta, \quad \forall (\theta, t) \in [-\tau, 0] \times [0, \infty).$$

Since the semiflow associated with (3.1) is monotone increasing on $\mathcal{C}_0$, we thus have

$$0 \leq 1 - v_\varepsilon(\alpha|\ln\varepsilon| + t + \theta) \leq 1 - V(\alpha|\ln\varepsilon| + t; \delta)(\theta),$$

which combined with (3.4) yields, for $\varepsilon > 0$ small enough,

$$0 \leq 1 - v_\varepsilon(\alpha|\ln\varepsilon| + t + \theta) \leq Me^{-\lambda(\alpha|\ln\varepsilon|+t)} \leq M\varepsilon^{\alpha\lambda} \leq \varepsilon^{\alpha\lambda/2}.$$

This completes the proof of Proposition 3.4. □

Derivatives of the semiflow. Let us now provide some estimates on the derivatives of the semiflow V_η with respect to the state variable. Our first result is a consequence of the well-known differentiability result of semiflows generated by delay differential equations (see for instance [10], see also [21] for results on abstract semilinear problems with Hille-Yosida non-densely defined operator).

Lemma 3.5 (Derivatives)**.** *For each $t > 0$, the map $\phi \in \mathcal{C}_0 \mapsto V_\eta(t;\phi) \in \mathcal{C}_0$ provided by Lemma 3.2 is of the class C^2. For each $\phi_0 \in \mathcal{C}_0$ and each $\phi \in \mathcal{C}_0$, the map $t \in [0,\infty) \mapsto \partial_\phi V_\eta(t;\phi_0)\cdot\phi \in \mathcal{C}_0$ is the mild solution of the non-autonomous equation*

$$\begin{cases} \dfrac{dv}{dt}(t) = L(t,\phi_0)v_t, \quad t > 0, \\ v(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{3.9}$$

wherein, for each $t > 0$, $L(t,\phi_0) : \mathcal{C}_0 \to \mathbb{R}$ is defined by

$$L(t,\phi_0)\phi := {f_\eta}'\left(V_\eta(t;\phi_0)(-\tau)\right)\phi(-\tau) - \phi(0). \tag{3.10}$$

Moreover, for each $\phi_0 \in \mathcal{C}_0$ and each $\phi \in \mathcal{C}_0$, the map $t \mapsto \partial^2_{\phi,\phi}V_\eta(t;\phi_0)\cdot(\phi,\phi)$ is the solution of

$$\begin{cases} \dfrac{dv}{dt}(t) = L(t,\phi_0)v_t + G(t;\phi_0;\phi), \quad t > 0, \\ v(\theta) = 0, \quad \theta \in [-\tau, 0], \end{cases} \tag{3.11}$$

wherein the map $t \mapsto G(t;\phi_0;\phi)$ is defined by

$$G(t;\phi_0;\phi) := {f_\eta}''\left(V_\eta(t;\phi_0)(-\tau)\right)\left[\partial_\phi V_\eta(t;\phi_0)\cdot\phi(-\tau)\right]^2. \tag{3.12}$$

Here is an estimate on the first derivative.

Lemma 3.6 (First derivative)**.** *There exist constants $M^+ > 1$ and $\gamma^+ > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$,*

$$e^{-\tau}e^{-(t+\theta)} \leq \partial_\phi V_\eta(t;\phi_0)\cdot 1(\theta) \leq M^+e^{\gamma^+(t+\theta)}, \quad \forall(\theta,t) \in [-\tau,0]\times[0,\infty).$$

Proof. Let $\phi_0 \in \mathcal{C}_0$ be given. First, the semiflow $V_\eta(t)$ being monotone increasing on $\mathcal{C}_0$, observe that

$$\partial_\phi V_\eta(t;\phi_0)\cdot 1(\theta) \geq 0, \quad \forall(\theta,t) \in [-\tau,0]\times[0,\infty). \tag{3.13}$$

Hence, in view of (3.9) and (3.10), the function $w(t) := \partial_\phi V_\eta(t;\phi_0)\cdot 1(0)$ satisfies

$$w'(t) \geq -w(t), \quad \forall t \geq 0,$$

so that $w(t) \geq e^{-t}$, for all $t \geq 0$, which in turn implies

$$\partial_\phi V_\eta(t;\phi_0)\cdot 1(\theta) \geq e^{-(t+\theta)},$$

for all $(\theta,t) \in [-\tau,0]\times[0,\infty)$ such that $t+\theta \geq 0$. For the remaining $(\theta,t) \in [-\tau,0]\times[0,\infty)$ such that $t+\theta < 0$, we have $\partial_\phi V_\eta(t;\phi_0)\cdot 1(\theta) = 1 \geq e^{-(\tau+t+\theta)}$. This completes the proof of the left-hand side of the estimate of the lemma.

Next, choosing a constant $\tilde{N} > 1$ such that

$$0 \leq {f_\eta}'(u) \leq \tilde{N}, \quad \forall u \in \mathbb{R}, \tag{3.14}$$

we infer from (3.9) and (3.10) that

$$w'(t) \leq \tilde{N}w(t-\tau) - w(t), \quad t > 0, \text{ and } w(\theta) = 1, \quad \theta \in [-\tau,0]. \tag{3.15}$$

Observe that the map $h : t \mapsto e^{(\tilde{N}-1)\tau}e^{(\tilde{N}-1)t}$ satisfies

$$h'(t) \geq \tilde{N}h(t-\tau) - h(t), \quad t > 0, \text{ and } h(\theta) \geq 1, \quad \theta \in [-\tau,0]. \tag{3.16}$$

It follows from (3.15) and (3.16) that $w(t) \leq e^{(\tilde{N}-1)\tau}e^{(\tilde{N}-1)t}$, for all $t \geq 0$. Arguing as above we get the right-hand side of the estimate of the lemma. □

We pursue with the following estimate on the second derivative.

Lemma 3.7 (Second derivative)**.** *There exist constants $K > 0$ and $\mu > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$,*

$$|\partial_{\phi\phi}V_\eta(t;\phi_0)\cdot(1,1)(\theta)| \leq Ke^{\mu(t+\theta)}, \quad \forall(\theta,t) \in [-\tau,0]\times[0,\infty).$$

Proof. In view of (3.12) and Lemma 3.6, there exists a constant $A > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$,

$$|G(t;\phi_0;1)| \leq Ae^{2\gamma^+(t-\tau)}, \quad \forall t \geq 0.$$

Hence, the function $w(t) := \partial_{\phi\phi}V_\eta(t;\phi_0)\cdot(1,1)(0)$ satisfies

$$w'(t) \leq \tilde{N}w(t-\tau) - w(t) + Ae^{2\gamma^+(t-\tau)}, \quad t > 0, \text{ and } w(\theta) = 0, \quad \theta \in [-\tau,0]. \tag{3.17}$$

We look for a super-solution of (3.17) in the form $t \mapsto \tilde{K}e^{\tilde{\mu}t}$, for some constants $\tilde{K} > 0$ and $\tilde{\mu} > 0$ to be determined. This leads us to

$$\tilde{\mu} \geq \tilde{N}e^{-\tilde{\mu}\tau} - 1 + \frac{A}{\tilde{K}}e^{-2\gamma^+\tau + (2\gamma^+ - \tilde{\mu})t}, \quad \forall t > 0, \tag{3.18}$$

which can be achieved by choosing $\tilde{\mu} > 2\gamma^+$ and $\tilde{K} > 0$ both large enough. Arguing as in the proof of Lemma 3.6, we end up with constants $K > 0$ and $\mu > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$, all $\theta \in [-\tau,0]$, all $t \geq 0$,

$$\partial_{\phi\phi}V_\eta\,(t;\phi_0)\cdot(1,1)(\theta) \leq Ke^{\mu(t+\theta)}.$$

Next, select $C > 0$ such that ${f_\eta}''(u) \geq -C$, for all $u \in \mathbb{R}$. Then we get $w'(t) \geq -Cw(t-\tau) - w(t) - Ae^{2\gamma^+(t-\tau)}$, for which we can construct a sub-solution $t \mapsto -\tilde{K}e^{\tilde{\mu}t}$ as above. This completes the proof of the lemma. □

As a direct consequence of Lemma 3.6 and Lemma 3.7, we obtain the following estimate.

Proposition 3.8 (Estimate on derivatives)**.** *There exist constants $\widehat{K} > 0$ and $\gamma > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$,*

$$|\partial_{\phi\phi} V_\eta(t;\phi_0) \cdot (1,1)(\theta)| \leq \widehat{K} e^{\gamma t} \partial_\phi V_\eta(t;\phi_0) \cdot 1(\theta),$$

for all $(\theta, t) \in [-\tau, 0] \times [0, \infty)$.

3.2 Construction of lower barriers for small times

We now provide an accurate lower estimate, for small times, of $u^\varepsilon : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to [0,1]$ the solution of (1.1)–(1.2).

Proposition 3.9 (Sub-solutions)**.** *Let the initial data φ satisfy Assumption 1.1 (i). Then there exist $K > 0$, $\alpha > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\max \left\{ 0\,; v_\eta \left(\frac{t}{\varepsilon}; w_0(x) - \varepsilon K\tau - Kt \right) \right\} \leq u^\varepsilon(t,x),$$

for all $(t,x) \in [-\varepsilon\tau, \alpha\varepsilon|\ln\varepsilon|] \times \mathbb{R}^N$. Here, $v_\eta = v_\eta(\cdot\,;\phi) : [-\tau, \infty) \to \mathbb{R}$ denotes the solution of (3.2) *arising in Lemma 3.2 and the function w_0 is as in* (1.6)*.*

Proof. Let us consider the differential operator

$$\mathcal{L}_\eta^\varepsilon[u](t,x) := \partial_t u(t,x) - \varepsilon \Delta u(t,x) - \frac{1}{\varepsilon}\big[f_\eta\left(u(t-\varepsilon\tau, x)\right) - u(t,x) \big].$$

Since $f_\eta = f$ on $[0,1]$, we have $\mathcal{L}_\eta^\varepsilon\left[u^\varepsilon\right](t,x) \equiv 0$. We look for a sub-solution, at least for small times, $\underline{u} : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$ in the form

$$\underline{u}(t,x) := v_\eta \left(\frac{t}{\varepsilon}; w_0(x) - \varepsilon K\tau - Kt \right).$$

Straightforward computations yield, for each $t > 0$ and each $x \in \mathbb{R}^N$,

$$\begin{aligned}
\mathcal{L}_\eta^\varepsilon\left[\underline{u}\right](t,x) &= -V^\varepsilon(t,x) \left[K + \varepsilon \Delta w_0(x) + \varepsilon \frac{W^\varepsilon(t,x)}{V^\varepsilon(t,x)} |\nabla w_0(x)|^2 \right] \\
&+ \frac{1}{\varepsilon} \left[\left(\frac{dv_\eta}{dt} + v_\eta \right) \left(\frac{t}{\varepsilon}; w_0(x) - \varepsilon\tau - Kt \right) - f_\eta \left(v_\eta \left(\frac{t}{\varepsilon} - \tau; w_0(x) - Kt \right) \right) \right]
\end{aligned}$$

where

$$\begin{aligned}
V^\varepsilon(t,x) &:= \partial_\phi V_\eta \left(\frac{t}{\varepsilon}; w_0(x) - \varepsilon K\tau - Kt \right) \cdot 1(0)\,, \\
W^\varepsilon(t,x) &:= \partial_{\phi\phi} V_\eta \left(\frac{t}{\varepsilon}; w_0(x) - \varepsilon K\tau - Kt \right) \cdot (1,1)(0)\,.
\end{aligned}$$

Since the semiflow arising in Lemma 3.2 is monotone increasing in $\mathcal{C}_0$ and since f_η is increasing, we have

$$\begin{aligned}
&\left(\frac{dv_\eta}{dt} + v_\eta \right) \left(\frac{t}{\varepsilon}; w_0(x) - \varepsilon K\tau - Kt \right) - f_\eta \left(v_\eta \left(\frac{t}{\varepsilon} - \tau; w_0(x) - Kt \right) \right) \\
&\leq \left(\frac{dv_\eta}{dt} + v_\eta \right) \left(\frac{t}{\varepsilon}; w_0(x) - \varepsilon K\tau - Kt \right) - f_\eta \left(v_\eta \left(\frac{t}{\varepsilon} - \tau; w_0(x) - \varepsilon K\tau - Kt \right) \right) \\
&= 0,
\end{aligned}$$

since v_η solves (3.2). Hence, using Proposition 3.8, we get, for all $\varepsilon \in (0,1)$, $t>0$, $x \in \mathbb{R}^N$,

$$\mathcal{L}_\eta^\varepsilon\left[\underline{u}\right](t,x) \leq -V^\varepsilon(t,x)\left[K - \varepsilon\|\Delta w_0\|_\infty - \varepsilon\|\nabla w_0\|_\infty^2 \widehat{K} e^{\gamma\frac{t}{\varepsilon}}\right].$$

Looking at small times, the above implies, for all $\varepsilon \in (0,1)$, $t \in \left(0, \gamma^{-1}\varepsilon|\ln\varepsilon|\right)$, $x \in \mathbb{R}^N$,

$$\mathcal{L}_\eta^\varepsilon\left[\underline{u}\right](t,x) \leq -V^\varepsilon(t,x)\left[K - \varepsilon\|\Delta w_0\|_\infty - \|\nabla w_0\|_\infty^2 \widehat{K}\right] \leq 0,$$

if $K>0$ is sufficiently large. Next, concerning initial data, we have, for all $\theta \in [-\varepsilon\tau, 0]$,

$$\underline{u}(\theta,x) = w_0(x) - \varepsilon K\tau - K\theta \leq w_0(x) \leq \varphi\left(\frac{\theta}{\varepsilon}, x\right) = u^\varepsilon(\theta,x),$$

where we have used (1.6) and (1.2). The comparison principle in Proposition 2.1 thus implies that

$$\underline{u}(t,x) \leq u^\varepsilon(t,x), \quad \forall (t,x) \in \left[-\varepsilon\tau, \gamma^{-1}\varepsilon|\ln\varepsilon|\right] \times \mathbb{R}^N.$$

Recalling that $u^\varepsilon \geq 0$, this completes the proof of Proposition 3.9. □

Proof of Proposition 3.1. Fix $K>0$ and $\alpha>0$ as in Proposition 3.9. Define $\alpha_0 := \alpha/2$. For $\phi :\equiv \alpha_0 \in \mathcal{C}_0 \setminus \{0\}$, let us select $\lambda > 0$ as in Proposition 3.4 and define $\rho_0 := \alpha_0\lambda/2$. Also, it follows from Assumption 1.1 (*ii*) that there exists $\delta_0 > 0$ such that, for $\varepsilon > 0$ small enough,

$$d(0,x) \leq -\delta_0\varepsilon|\ln\varepsilon| \Longrightarrow w_0(x) \geq 4\alpha_0\varepsilon|\ln\varepsilon|. \tag{3.19}$$

Now, for any $-\tau \leq \theta \leq 0$, define $s := \alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau + \varepsilon\theta$ and take x such that $d(0,x) \leq -\delta_0\varepsilon|\ln\varepsilon|$. Since, for $\varepsilon>0$ small enough, $0 \leq s \leq \alpha\varepsilon|\ln\varepsilon|$ and $w_0(x) - \varepsilon K\tau - Ks \geq \alpha_0\varepsilon|\ln\varepsilon|$, we deduce from Proposition 3.9 and Proposition 3.4 that

$$u^\varepsilon(s,x) \geq v_\eta(\alpha_0|\ln\varepsilon| + \tau + \theta; \alpha_0\varepsilon|\ln\varepsilon|) \geq 1 - \varepsilon^{\rho_0},$$

which concludes the proof. □

4 Lower barriers via bistable approximation

As explained before, our analysis of the propagation of interface from below is performed by approximating the monostable function f in a bistable manner (see subsection 2.2). We start with some preliminaries on smooth signed distance functions associated with a family of free boundary problems.

4.1 Smooth cut-off signed distance functions

For $c>0$, we denote by $\Gamma^c := \bigcup_{t\geq 0}(\{t\} \times \Gamma_t^c)$ the smooth solution of the free boundary problem

$$(P^c) \quad \begin{cases} V = c & \text{on } \Gamma_t^c \\ \Gamma_t^c\big|_{t=0} = \Gamma_0, \end{cases}$$

where V denotes the normal velocity of Γ_t^c in the exterior direction. Note that since the region enclosed by Γ_0, namely Ω_0, is convex, these solutions do exist for all $t \geq 0$. Also we can naturally, i.e. in a *reversible* manner, extend these solutions for small negative times by letting Γ_0 evolve with speed $-c$. Hence, with a slight abuse of notation, we consider Γ_t^c for all $t \geq -\varepsilon\tau$, with $\varepsilon > 0$ small enough. For each $t \geq -\varepsilon\tau$, we denote by Ω_t^c the region enclosed by the hypersurface Γ_t^c.

Let $\widetilde{d}$ be the signed distance function to Γ^c defined by

$$\widetilde{d}(t,x) := \begin{cases} -\text{dist}(x, \Gamma_t^c) & \text{for } x \in \Omega_t^c \\ \ \ \text{dist}(x, \Gamma_t^c) & \text{for } x \in \mathbb{R}^N \setminus \Omega_t^c, \end{cases} \tag{4.1}$$

where $\text{dist}(x, \Gamma_t^c)$ is the distance from x to the hypersurface Γ_t^c. We remark that $\widetilde{d} = 0$ on Γ^c and that $|\nabla \widetilde{d}| = 1$ in a neighborhood of Γ^c.

We now introduce the "cut-off signed distance function" d, which is defined as follows. Let $T > 0$ be given. First, choose $d_0 > 0$ small enough so that $\widetilde{d}$ is smooth in the tubular neighborhood of Γ^c

$$\{(t,x) \in [-\varepsilon\tau, T] \times \mathbb{R}^N : \ |\widetilde{d}(t,x)| < 3d_0\}.$$

Next let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leq d_0 \\ -2d_0 & \text{if } s \leq -2d_0 \\ 2d_0 & \text{if } s \geq 2d_0. \end{cases}$$

We then define the cut-off signed distance function d by

$$d(t,x) := \zeta\left(\tilde{d}(t,x)\right). \tag{4.2}$$

Note that

$$\text{if} \quad |d(t,x)| < d_0 \quad \text{then} \quad |\nabla d(t,x)| = 1, \tag{4.3}$$

and that the equation of motion (P^c) yields

$$\text{if} \quad |d(t,x)| < d_0 \quad \text{then} \quad \partial_t d(t,x) + c = 0. \tag{4.4}$$

Then the mean value theorem provides a constant $\bar{N} > 0$ such that

$$|\partial_t d(t,x) + c| \leq \bar{N}|d(t,x)| \quad \text{for all } (t,x) \in [-\varepsilon\tau, T] \times \mathbb{R}^N. \tag{4.5}$$

Moreover, there exists a constant $C > 0$ such that

$$|\nabla d(t,x)| + |\Delta d(t,x)| \leq C \quad \text{for all } (t,x) \in [-\varepsilon\tau, T] \times \mathbb{R}^N. \tag{4.6}$$

4.2 Construction of lower barriers

Let us recall that $\{f_\eta\}_{\eta \in (0,1]}$ denotes a family of bistable approximations of f such that (2.3) and (2.4) hold. Also, for $\eta \in (0,1]$, (U_η, c_η) denotes the travelling wave solution (with time delay) associated with this bistable f_η (see Lemma 2.2), namely

$$\begin{cases} U_\eta''(z) + c_\eta U_\eta'(z) + f_\eta\left(U_\eta(z + c_\eta\tau)\right) - U_\eta(z) = 0, \quad \forall z \in \mathbb{R}, \\ U_\eta(-\infty) = 1, \quad U_\eta(0) = 0, \quad U_\eta(\infty) = -\eta. \end{cases} \tag{4.7}$$

In the spirit of the sub-solutions constructed in [3] for bistable systems, we look for sub-solutions u_η^- in the form

$$u_\eta^-(t,x) := U_\eta\left(\frac{d_\eta(t,x)+\varepsilon|\ln\varepsilon|p(t)}{\varepsilon}\right) - q(t), \tag{4.8}$$

where

$$\begin{aligned} p(t) &:= -e^{-\beta t/\varepsilon} + e^{Lt} + K, && (4.9)\\ q(t) &:= \sigma\left(\beta e^{-\beta t/\varepsilon} + \varepsilon L e^{Lt}\right). && (4.10)\end{aligned}$$

Here, σ, β, L and K are positive constants to be determined, and $d_\eta(t,x)$ denotes the cut-off signed distance function to the interface starting from Γ_0 and evolving with speed c_η, that is the solution of (P^{c_η}). As seen in the previous subsection, this allows to define u_η^- for all $t \geq -\varepsilon\tau$, $x \in \mathbb{R}^N$.

Proposition 4.1 (Sub-solutions). *One can find positive constants β, σ and L such that, for all $K > 1$, the function u_η^- satisfies, for $\varepsilon > 0$ small enough,*

$$\varepsilon\mathcal{L}_\eta^\varepsilon[u_\eta^-](t,x) = \varepsilon\partial_t u_\eta^-(t,x) - \varepsilon^2\Delta u_\eta^-(t,x) - f_\eta\left(u_\eta^-(t-\varepsilon\tau,x)\right) + u_\eta^-(t,x) \leq 0,$$

for all $t > 0$, $x \in \mathbb{R}^N$.

Proof. For ease of notation, we drop most of the subscripts η. Also we define

$$z := \frac{d(t,x)+\varepsilon|\ln\varepsilon|p(t)}{\varepsilon}. \tag{4.11}$$

We start by evaluating $\varepsilon\mathcal{L}_\eta^\varepsilon[u^-](t,x)$. We compute

$$\begin{aligned}\varepsilon\partial_t u^-(t,x) &= (\partial_t d(t,x) + \varepsilon|\ln\varepsilon|p'(t))U'(z) - \varepsilon q'(t)\\ \varepsilon^2\Delta u^-(t,x) &= |\nabla d|^2(t,x)U''(z) + \varepsilon\Delta d(t,x)U'(z).\end{aligned}$$

Next, observe that the previous subsection enables to write

$$d(t-\varepsilon\tau,x) = d(t,x) + \varepsilon c\tau + \varepsilon\Theta_\varepsilon(t,x),$$

where the correction Θ_ε vanishes close to the interface and is $\mathcal{O}(1)$:

$$\Theta_\varepsilon(t,x) = 0 \quad \text{if } |d(t,x)| \leq d_0, \quad \|\Theta_\varepsilon\|_{L^\infty} \leq A, \tag{4.12}$$

for some constant $A > 0$. Hence, since $p(t)$ increases and $U(z)$ decreases, we have

$$\begin{aligned} u^-(t-\varepsilon\tau,x) &= U\left(\frac{d(t,x)+\varepsilon|\ln\varepsilon|p(t-\varepsilon\tau)}{\varepsilon} + c\tau + \Theta_\varepsilon(t,x)\right) - q(t-\varepsilon\tau)\\ &\geq U\left(\frac{d(t,x)+\varepsilon|\ln\varepsilon|p(t)}{\varepsilon} + c\tau + \Theta_\varepsilon(t,x)\right) - q(t-\varepsilon\tau).\end{aligned}$$

Since f is increasing we get

$$\begin{aligned} f\left(u^-(t-\varepsilon\tau,x)\right) &\geq f\Big(U\left(z + c\tau + \Theta_\varepsilon(t,x)\right) - q(t-\varepsilon\tau)\Big)\\ &= f\Big(U(z + c\tau + \Theta_\varepsilon(t,x))\Big) - q(t-\varepsilon\tau)f'(\theta),\end{aligned}$$

for some $U(z+c\tau+\Theta_\varepsilon(t,x))-q(t-\varepsilon\tau)\leq\theta\leq U(z+c\tau)$. Hence, we have

$$
\begin{aligned}
f\left(u^-(t-\varepsilon\tau,x)\right) \quad \geq \quad & f\left(U(z+c\tau)\right)-q(t-\varepsilon\tau)f'(\theta) \\
& +\Theta_\varepsilon(t,x)(f\circ U)'\left(z+c\tau+\omega\Theta_\varepsilon(t,x)\right),
\end{aligned}
$$

for some $0\leq\omega\leq 1$. Combining the above estimates with $U''(z)+cU'(z)+f(U(z+c\tau))-U(z)=0$, we obtain $\varepsilon\mathcal{L}^\varepsilon_\eta[u^-](t,x)\leq E_1+E_2+E_3$ where

$$
\begin{aligned}
E_1 &:= \varepsilon|\ln\varepsilon|p'(t)U'(z)+q(t-\varepsilon\tau)f'(\theta)-q(t)-\varepsilon q'(t) \\
E_2 &:= (\partial_t d(t,x)+c-\varepsilon\Delta d(t,x))U'(z)+\left(1-|\nabla d(t,x)|^2\right)U''(z) \\
E_3 &:= -\Theta_\varepsilon(t,x)(f\circ U)'\left(z+c\tau+\omega\Theta_\varepsilon(t,x)\right).
\end{aligned}
$$

Let us now analyze further the term E_1. By using the expressions (4.9), (4.10) for p and q we obtain

$$
\begin{aligned}
E_1 \quad &= \quad \beta e^{-\beta t/\varepsilon}\left(|\ln\varepsilon|U'(z)+\sigma(e^{\beta\tau}f'(\theta)-1+\beta)\right) \\
&\qquad +\varepsilon Le^{Lt}\left(|\ln\varepsilon|U'(z)+\sigma(e^{-\varepsilon L\tau}f'(\theta)-1-\varepsilon L)\right) \\
&=: \quad \beta e^{-\beta t/\varepsilon}I_1+\varepsilon Le^{Lt}I_2.
\end{aligned}
$$

Since $f'(-\eta)<1$ and $f'(1)<1$, we can fix small $a>0$ and $\beta>0$ such that

$$
e^{\beta\tau}f'(u)-1+\beta\leq-\beta, \quad \forall u\in[-\eta-a,-\eta+a]\cup[1-a,1+a].
$$

In view of $U(-\infty)=1$, $U(\infty)=-\eta$ and inequality $U(z+c\tau+\Theta_\varepsilon(t,x))-q(t-\varepsilon\tau)\leq\theta\leq U(z+c\tau)$, there exists a large z_0 such that $\theta\in[-\eta-a,-\eta+a]\cup[1-a,1+a]$ as soon as $|z|\geq z_0$ (by choosing σ small enough to control the $-q(t-\varepsilon\tau)$ term) and the above inequality applies for $s=\theta$. It follows from $U'(z)\leq 0$ that $I_1\leq-\sigma\beta$ in the region $\{|z|\geq z_0\}$. In the compact region $\{|z|\leq z_0\}$, we have $U'(z)\leq -b$ for some $b>0$ so that $I_1\leq -b|\ln\varepsilon|+C$ so that $I_1\leq-\sigma\beta$ also holds true. The same argument yields $I_2\leq-\sigma\beta$. Hence

$$
E_1\leq-\sigma\beta^2e^{-\beta t/\varepsilon}-\varepsilon\sigma\beta Le^{Lt}\leq-\varepsilon\sigma\beta L.
$$

We now conclude the proof of $\varepsilon\mathcal{L}^\varepsilon_\eta[u^-](t,x)\leq 0$. Assume first that (t,x) lies in the tubular neighborhood $\{|d(t,x)|\leq d_0\}$ of Γ_t. In view of (4.3) and (4.4), the term E_2 reduces to $-\varepsilon\Delta d(t,x)U'(z)$. In view of (4.12), the term E_3 vanishes. As a result,

$$
\varepsilon\mathcal{L}^\varepsilon_\eta[u^-](t,x)\leq-\varepsilon\sigma\beta L+\varepsilon\|\Delta d\|_{L^\infty}\|U'\|_{L^\infty(\mathbb{R})}\leq 0,
$$

if $L>0$ is large enough. Next, if (t,x) is such that $|d(t,x)|\geq d_0$ then we shall use the exponential decay of the derivatives of U — see Lemma 2.2 (ii)— to control E_2 and E_3. Indeed in this region, the argument z defined in (4.11) satisfies $|z|\geq d_0/(2\varepsilon)$. Hence, combining the exponential decay of U' and U'' with (4.5) and (4.6), we get a bound $|E_2|\leq C_2e^{-C_2\frac{d_0}{2\varepsilon}}$, for some $C_2>0$. Also, it follows from (4.12) that

$$
|z+c\tau+\omega\Theta_\varepsilon(t,x)|\geq\frac{d_0}{2\varepsilon}-c\tau-\omega A\geq\frac{d_0}{4\varepsilon},
$$

which in turn provides a bound $|E_3| \leq C_3 e^{-C_3 \frac{d_0}{4\varepsilon}}$, for some $C_3 > 0$. As a result we collect, for a constant $C > 0$,

$$\varepsilon \mathcal{L}^{\varepsilon}_{\eta}[u^-](t,x) \leq -\varepsilon\sigma\beta L + Ce^{-C\frac{d_0}{4\varepsilon}} \leq 0,$$

if $\varepsilon > 0$ is small enough. This completes the proof of the lemma. □

In order to apply the comparison principle, we need the following estimate.

Lemma 4.2 (Ordering initial data). *One can find $K > 1$ such that, for $\varepsilon > 0$ small enough,*

$$u^-_\eta(t,x) \leq u^\varepsilon(t + \alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau, x), \qquad \textit{for all } -\varepsilon\tau \leq t \leq 0,\ x \in \mathbb{R}^N,$$

where $\alpha_0\varepsilon|\ln\varepsilon|$ denotes the "generation of interface from below time" appearing in Proposition 3.1.

Proof. For ease of notation, we drop most of the subscripts η. If (t,x) is such that $d(t,x) \geq -\varepsilon|\ln\varepsilon|p(t)$, then the decrease of the wave U yields $u^-(t,x) \leq 0$, and there is nothing to prove. Now let us take (t,x), with $-\varepsilon\tau \leq t \leq 0$ and $d(t,x) \leq -\varepsilon|\ln\varepsilon|p(t)$. From the generation of interface from below analysis we know that (see Proposition 3.1)

$$d(0,x) \leq -\delta_0\varepsilon|\ln\varepsilon| \Longrightarrow 1 - \varepsilon^{\rho_0} \leq u^\varepsilon(\alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau + t, x) \quad \text{for } -\varepsilon\tau \leq t \leq 0. \tag{4.13}$$

Writing $d(0,x) = d(t,x) + \mathcal{O}(t)$ and using the expression for p in (4.9), we get, for $-\varepsilon\tau \leq t \leq 0$,

$$\begin{aligned} d(0,x) &\leq -\varepsilon|\ln\varepsilon|p(t) + C\varepsilon\tau \\ &\leq -\varepsilon|\ln\varepsilon|(-e^{\beta\tau} + e^{-\varepsilon L\tau} + K) + C\varepsilon\tau \\ &\leq -\delta_0\varepsilon|\ln\varepsilon|, \end{aligned}$$

for $\varepsilon > 0$ small enough, if K is chosen sufficiently large. In view of (4.13) it suffices to show that $u^-(t,x) \leq 1 - \varepsilon^{\rho_0}$, which follows from the vertical shift q. Indeed, the expression for q in (4.10) shows that $q(t) \geq \sigma\beta$ for $-\varepsilon\tau \leq t \leq 0$, so that $u^-(t,x) \leq 1 - \sigma\beta \leq 1 - \varepsilon^{\rho_0}$. The lemma is proved. □

Proof of Theorem 1.3 (i)**.** From Proposition 4.1, Lemma 4.2 and the comparison principle, we infer that

$$u^-_\eta(t - \alpha_0\varepsilon|\ln\varepsilon| - \varepsilon\tau, x) \leq u^\varepsilon(t,x) \quad \text{for all } t \geq \alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau,\ x \in \mathbb{R}^N. \tag{4.14}$$

Let us recall that u^-_η is defined in (4.8) and that $U_\eta(-\infty) = 1$. Hence, the convergence to 1 in $\Omega_t^{c^*}$, as expressed in Theorem 1.3 (i), is a direct consequence of both Lemma 2.4 and the lower estimate (4.14). □

5 Global in time upper barriers

The aim of this section is to construct a super-solution in order to control the propagation of the solution from above. Let (U^*, c^*) be the monostable travelling wave with the minimal speed $c^* > 0$ (see Lemma 2.3), namely

$$\begin{cases} (U^*)''(z) + c^*(U^*)'(z) + f\left(U^*(z + c^*\tau)\right) - U^*(z) = 0, \ \ \forall z \in \mathbb{R}, \\ (U^*)'(z) < 0, \ \ \forall z \in \mathbb{R}, \\ U^*(-\infty) = 1 \ \text{ and } \ U^*(\infty) = 0. \end{cases}$$

Then we shall prove the upper estimate on $u^\varepsilon : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to [0, 1]$ the solution of (1.1)–(1.2).

Proposition 5.1 (Super-solutions)**.** *Let the initial data φ satisfy Assumption 1.1. Denote by $d(0, x)$ the smooth cut-off signed distance function to Γ_0 as defined in subsection 4.1 (in particular, $d(0, x) < 0$ if and only if $x \in \Omega_0$). Then there exists $h \in \mathbb{R}$ such that, for all $\varepsilon > 0$ small enough,*

$$u^\varepsilon(t, x) \leq U^* \left(\frac{d(0, x) - c^*t}{\varepsilon} + h \right), \ \ \forall (t, x) \in [-\varepsilon\tau, \infty) \times \mathbb{R}^N.$$

Proof. Since the function v_0 appearing in Assumption 1.1 (iii) satisfies $\|v_0\|_\infty < 1$, we can choose $h \in \mathbb{R}$ such that $\|v_0\|_\infty \leq U^*(c^*\tau + h)$. Up to changing U^* by $U^*(\cdot + h)$, we can assume $h = 0$ so that

$$\|v_0\|_\infty \leq U^*(c^*\tau). \tag{5.1}$$

Let $x_0 \in \partial\Omega_0 = \Gamma_0$ be given and denote by n_0 the outward unit normal vector to Γ_0 at x_0. Then consider the map $u^+ : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$u^+(t, x) := U^* \left(\frac{(x - x_0).n_0 - c^*t}{\varepsilon} \right).$$

Setting $z = \dfrac{(x - x_0).n_0 - c^*t}{\varepsilon}$, we compute

$$\begin{aligned} \mathcal{L}^\varepsilon[u^+](t, x) &:= \partial_t u^+(t, x) - \varepsilon\Delta u^+(t, x) - \frac{1}{\varepsilon} f\left(u^+(t - \varepsilon\tau, x)\right) + \frac{1}{\varepsilon} u^+(t, x) \\ &= -\frac{c^*}{\varepsilon} (U^*)'(z) - \frac{1}{\varepsilon} (U^*)''(z) - \frac{1}{\varepsilon} f\left(U^*(z + c^*\tau)\right) + \frac{1}{\varepsilon} U^*(z) \\ &= 0, \end{aligned}$$

for all $t > 0$, $x \in \mathbb{R}^N$. Let us now prove that

$$u^\varepsilon(\theta, x) = \varphi\left(\frac{\theta}{\varepsilon}, x\right) \leq U^* \left(\frac{(x - x_0).n_0 - c^*\theta}{\varepsilon} \right) = u^+(\theta, x),$$

for all $(\theta, x) \in [-\varepsilon\tau, 0] \times \mathbb{R}^N$. In view od Assumption 1.1 (iii) and the decrease of U^*, it is sufficient to check that

$$v_0(x) \leq U^* \left(\frac{(x - x_0).n_0}{\varepsilon} + c^*\tau \right), \ \ \forall x \in \mathbb{R}^N. \tag{5.2}$$

When $(x - x_0).n_0 \leq 0$, the above inequality follows from (5.1). When $(x - x_0).n_0 > 0$, (1.8) and the convexity of Ω_0 implies $v_0(x) = 0$ and (5.2) is clear. Hence, it follows from the comparison principle that

$$u^\varepsilon(t, x) \leq U^* \left(\frac{(x - x_0).n_0 - c^*t}{\varepsilon} \right), \ \ \forall (t, x) \in [-\varepsilon\tau, \infty) \times \mathbb{R}^N,$$

for each $x_0 \in \partial\Omega_0$. This completes the proof of the proposition. □

Remark 5.2. If $\|v_0\|_\infty = 1$ then, under assumption (1.10) of Remark 1.2, we have $\mathcal{L}^\varepsilon[K_0 u^+](t,x) \geq 0$. Also, normalizing the travelling wave U^* by $1 = K_0 U^*(c^*\tau)$ and arguing as above, we see that $u^\varepsilon(\theta,x) \leq K_0 u^+(\theta,x)$, for all $(\theta,x) \in [-\varepsilon\tau, 0] \times \mathbb{R}^N$. Hence, the comparison principle yields

$$u^\varepsilon(t,x) \leq K_0 U^* \left(\frac{(x-x_0).n_0 - c^* t}{\varepsilon} \right), \quad \forall (t,x) \in [-\varepsilon\tau, \infty) \times \mathbb{R}^N,$$

for each $x_0 \in \partial\Omega_0$.

Proof of Theorem 1.3 ***(ii)*****.** The convergence to 0 outside $\Omega_t^{c^*}$, as expressed in Theorem 1.3 *(ii)*, is a direct consequence of the control from above provided by Proposition 5.1. □

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