# Review and Classification of Branching Schemes for Branch-and-Price

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# Dantzig-Wolfe reformulation

$$(IP) \qquad z = \min\{cx : \underbrace{Dx \geq d, Bx \geq b, x \in \mathbb{Z}^n_+}_{x \in X}\}$$

where $Dx \geq d$ represent **"complicating constraints"** while the set $Z = \{x \in \mathbb{Z}^n_+ : Bx \geq b\}$ is **"more tractable"**

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- Relaxing $Dx \geq d$ while penalizing (pricing) their violation in the objective $\rightarrow$ **Lagrangian relaxation**

$$\begin{aligned} L(\pi) &= \min_x\{cx + \pi(d - Dx) : Bx \geq b, x \in \mathbb{Z}^n_+\} \\ z_{LD} &= \max_{\pi \geq 0} L(\pi). \end{aligned}$$

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- Relaxing $Dx \geq d$ while penalizing (pricing) their violation in the objective $\rightarrow$ **Lagrangian relaxation**

- Reformulate the problem as selecting a solution from $Z$ that satisfy $Dx \geq d$ $\rightarrow$ **Dantzig-Wolfe Reformulation**

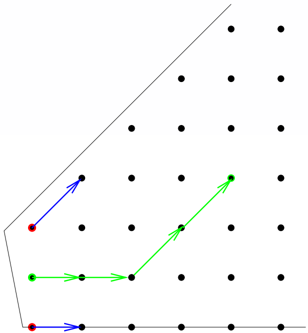
# Dantzig-Wolfe reformulation

## IP-Reformulation of $Z$ $\rightarrow$ discretization

Every IP set $Z = \{x \in \mathbb{Z}^n : Bx \geq b\}$ can be represented in the form $Z$ =proj$_x(Q)$, with

$$Q = \{(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{Z}_+^{|G|} \times \mathbb{Z}_+^{|R|} :$$
$$x = \sum_{g \in G} \lambda_g x^g + \sum_{r \in R} \mu_r v^r, \sum_{g \in G} \lambda_g = 1\},$$

where $\{x^g\}_{g \in G}$ is a finite set of integer points in $Z$, and $\{v^r\}_{r \in R}$ are the extreme integer rays of *conv*$(Z)$.



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where $\{x^g\}_{g \in G}$ is a finite set of integer points in $Z$, and $\{v^r\}_{r \in R}$ are the extreme integer rays of *conv*$(Z)$.

- Single Sub-system (assuming $Z$ is bounded):

$$\begin{array}{rcl} \min \sum_{g \in G} (cx^g) \lambda_g & & \\ \sum_{g \in G} (Dx^g) \lambda_g & \geq & d \\ \sum_{g \in G} \lambda_g & = & 1 \\ \lambda_g & \in & \{0, 1\} \; \forall g \in G \end{array}$$

# Dantzig-Wolfe reformulation

## The block diagonal case

$$
\begin{array}{lccccccccc}
\min & c^1 x^1 & + & c^2 x^2 & + & \dots & + & c^K x^K & & \\
 & D^1 x^1 & + & D^2 x^2 & + & \dots & + & D^K x^K & \geq & d \\
 & B^1 x^1 & & & & & & & \geq & b^1 \\
 & & & B^2 x^2 & & & & & \geq & b^2 \\
 & & & & & \ddots & & & \geq & \vdots \\
 & & & & & & & B^K x^K & \geq & b^K \\
 & x^1 & \in \mathbb{Z}_+^{n_1}, & x^2 & \in \mathbb{Z}_+^{n_2}, & \dots & x^K & \in \mathbb{Z}_+^{n_K}. & &
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\end{array}
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Relaxing $Dx \geq d \rightarrow K$ sub-problems:

$$\min\{c^k x^k : \underbrace{B^k x^k \geq b^k, x^k \in \mathbb{Z}_+^n}_{x^k \in Z^k}\}$$

# Dantzig-Wolfe reformulation

- Multiple Non-identical Sub-systems:

$$
\begin{aligned}
\min \textstyle\sum_{k=1}^{K} \sum_{g \in G^k} c^k\, x^g\, \lambda_g^k & \\
\textstyle\sum_{k=1}^{K} \sum_{g \in G^k} D^k\, x^g\, \lambda_g^k &\geq d \\
\textstyle\sum_{g \in G^k} \lambda_g^k &= 1 \;\; \forall\, k \\
\lambda_g^k &\in \{0, 1\} \;\; \forall\, k, g \in G^k
\end{aligned}
$$

- Multiple Identical Sub-systems:

$$\nu_g = \textstyle\sum_{k=1}^{K} \lambda_g^k \, .$$

$$
\begin{aligned}
\min \textstyle\sum_{g \in G} c\, x^g\, \nu_g & \\
\textstyle\sum_{g \in G} D\, x^g\, \nu_g &\geq d \\
\textstyle\sum_{g \in G} \nu_g &= K \\
\nu_g &\in I\!N \;\; \forall g \in G
\end{aligned}
$$

The "complicating" constraints only depend on the aggregate variables:

$$\boldsymbol{y = \textstyle\sum_{k=1}^{K} x^k = \sum_{g \in G} x^g \nu_g}.$$

# Dantzig-Wolfe reformulation

## Interests

1] To obtain better LP bounds

### Lagrangian duality

$$\begin{array}{rcl} z_{LD} & = & \min\{cx : Dx \geq d, x \in \mathit{conv}(Z)\} \\ \Updownarrow & & \\ z_{DW} & = & \min \sum_{g \in G} cx^g \lambda_g \\ & & \sum_{g \in G} Dx^g \lambda_g \geq d \\ & & \sum_{g \in G} \lambda_g = 1,\ \lambda_g \geq 0 \quad g \in G \end{array}$$

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2] To eliminate symmetry in the presence of identical subsystems (Be aware: it re-appears when branching).

3] To lead to a decomposition approach (specific oracle)

# Modeling integrality

$$\begin{array}{rcl} \min \sum_{g\in G}(cx^g)\lambda_g & & \\ \sum_{g\in G}(Dx^g)\lambda_g & \geq & d \\ \sum_{g\in G}\lambda_g & = & 1 \\ \lambda_g & \geq & 0 \ \forall g \in G \end{array}$$

- Single Sub-system (or Non-identical Sub-systems):
  - $x = \sum_{g\in G} x^g \lambda_g \in \mathbb{Z}^n$ **(projection)**
  - $\lambda_g \in \{0,1\} \ \forall g \in G$

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(3) MIP Extension: $x = (x_I, x_C)$, $\sum_{g\in G(x_I^g)} \lambda_g \in \{0,1\}$. **[Van07]**

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- Identical Sub-systems:
  - $\nu_g \in I\!N \;\forall g \in G$ (2), (3)

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# Checking integrality

- Single Sub-system (or Non-identical Sub-systems):
  - NSC: $x_i = \sum_{g \in G} x_i^g \lambda_g \in \mathbb{Z} \quad \forall i$
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  - SC2 **(disaggregation $\nu_g \to \lambda_g^k$ / projection $\lambda_g^k \to x^k$)**:

| $\nu_g$ | 1 | 0 | 1 | $\mathbf{\frac{1}{2}}$ | 0 | 0 | $\mathbf{\frac{1}{2}}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_{i_1}$ | 1 | 1 | 1 | **1** | 1 | 1 | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_{i_2}$ | 1 | 1 | 1 | **0** | 0 | 0 | **0** | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $x_{i_3}$ | 1 | 0 | 0 | **1** | 1 | 0 | **0** | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| $x_{i_4}$ | 0 | 1 | 0 | **1** | 0 | 1 | **0** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | $x^{k_1}$ | $x^{k_2}$ | | | $x^{k_3}$ | | | $x^{k_4}$ | | | | $x^{k_5}$ | | | |

$$\Rightarrow \boxed{\boldsymbol{x_{i_3}^{k_3} = \frac{1}{2}}}$$

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  - SC2 **(disaggregation $\nu_g \rightarrow \lambda_g^k$ / projection $\lambda_g^k \rightarrow x^k$)**:
    - lexicographic ordering of the columns
    - decomposition:
      $\lambda_g^k := \min\{1, \nu_g - \sum_{\kappa=1}^{k-1} \lambda_g^\kappa, (k - \sum_{\gamma \prec g} \nu_g)^+\} \quad \forall k, g \in G$
    - integrality test: $x_i^k = \sum_{g \in G} x_i^g \lambda_g^k \in \mathbb{Z} \quad \forall i$

# Branching

## Single subsystem (or multiple non-identical subsystems)

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- Branching on variable of the Dantzig-Wolfe reform.:
  - Option 0: Select $\lambda_g^* \notin \{0,1\}$ and set $\lambda_g \leq 0$ or $\lambda_g \geq 1$.
    - unbalance tree.
    - destroys subproblem structure to enforce $\lambda_g \leq 0$.

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- Branching on a variable of the original formulation:
  Select $x_j$ for which $x_j^* = \sum_{g\in G} x_j^g \lambda_g \notin \mathbb{Z}$. Separate into $x_j \leq \lfloor x_j^* \rfloor$ or $x_j \geq \lceil x_j^* \rceil$.

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  $$x_j \leq \lfloor x_j^* \rfloor \text{ or } x_j \geq \lceil x_j^* \rceil.$$
  - Option 1: The branching constraint is dualized as a "difficult" constraint. ("soft" branching constraint)
  - Option 2: The branching constraint is enforced in the sub-problem. ("hard" branching constraint)

# Branching

## Single subsystem: consider the up-branch

- Option 1: The branching constraint goes in the master:

$$\begin{aligned} \min \sum_{g \in G} (cx^g)\lambda_g & & \\ \sum_{g \in G} (Dx^g)\lambda_g &\geq d \\ \sum_{g \in G} x_j^g \lambda_g &\geq \lceil x_j^* \rceil \\ \sum_{g \in G} \lambda_g &= 1 \\ \lambda_g &\geq 0 \quad g \in G, \end{aligned}$$

- Option 2: The branching constraint goes in the subproblem:

$$\zeta_2 = \min\{(c - \pi D)x : x \in Z \cap \{x : x_j \geq \lceil x_j^* \rceil\}\}.$$

# Branching

## Single subsystem: comparing Option 1 & Option 2

### Strength of the linear programming bound

$$\begin{aligned} z^{MLP_1} &= \min\{cx : Dx \geq d, x \in \mathit{conv}(Z), x_j \geq \lceil x_j^* \rceil\} \\ \leq z^{MLP_2} &= \min\{cx : Dx \geq d, x \in \mathit{conv}(Z \cap \{x : x_j \geq \lceil x_j^* \rceil\}) \end{aligned}$$

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**Complexity of the subproblem** For option 1 the subproblem is unchanged, whereas for option 2 the subproblem may become more complicated.

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**Complexity of the subproblem** For option 1 the subproblem is unchanged, whereas for option 2 the subproblem may become more complicated.

**Getting Integer Solutions** If an optimal solution $x^*$ of IP is not an extreme point of *conv*$(Z)$, it cannot be obtained as a solution of the subproblem under Option 1. Under Option 2, one can eventually generate a column $x^g = x^*$ in the interior of $conv(Z)$.

## Multiple identical subsystems : the set partitioning case

$\nu_g = \sum_k \lambda_g^k$

$$\begin{aligned} \min \sum_{g \in G} (cx^g)\nu_g & & \\ \sum_{g \in G} x_i^g \nu_g &= 1 \ \forall i \\ \sum_{g \in G} \nu_g &= K \\ \nu_g &\geq 0 \ \forall g \in G \end{aligned}$$

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- Branching on an original var. in the disaggregated form.: $\boldsymbol{x_i^k = \sum_g x_i^g \lambda_g^k}$

**[Vil05]**

$$
\begin{array}{rcl}
\min \sum_{k=1}^K \sum_{g \in G^k} c^k \ x^g \ \lambda_g^k & & \\
\sum_{k=1}^K \sum_{g \in G^k} D^k \ x^g \ \lambda_g^k & \geq & d \\
\sum_{g \in G^k} \lambda_g^k & = & 1 \ \ \forall \ k \\
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## Multiple identical subsystems : the set partitioning case

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- Branching on an original var. in the disaggregated form.: $x_i^k = \sum_g x_i^g \lambda_g^k$

**[Vil05]**

$$\begin{array}{rcl} \min \sum_{k=1}^K \sum_{g \in G^k} c^k \, x^g \, \lambda_g^k & & \\ \sum_{k=1}^K \sum_{g \in G^k} D^k \, x^g \, \lambda_g^k & \geq & d \\ \sum_{g \in G^k} \lambda_g^k & = & 1 \ \forall \, k \\ \lambda_g^k & \in & \{0,1\} \ \forall \, k, g \in G^k \end{array}$$

- Branching on a aggregate original var.: $y_i = \sum_{g \in G} x_i^g \nu_g$ is **not an option** as $y_i = 1$ in all master LP solutions.

# Branching

## Multiple identical subsystems : the set partitioning case

$\nu_g = \sum_k \lambda_g^k$

$$\begin{array}{rcl} \min \sum_{g \in G} (cx^g)\nu_g & & \\ \sum_{g \in G} x_i^g \nu_g & = & 1 \ \forall i \\ \sum_{g \in G} \nu_g & = & K \\ \nu_g & \geq & 0 \ \forall g \in G \end{array}$$

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$$\begin{array}{rcl} \min \sum_{k=1}^{K} \sum_{g \in G^k} c^k \ x^g \ \lambda_g^k & & \\ \sum_{k=1}^{K} \sum_{g \in G^k} D^k \ x^g \ \lambda_g^k & \geq & d \\ \sum_{g \in G^k} \lambda_g^k & = & 1 \ \ \forall \ k \\ \lambda_g^k & \in & \{0, 1\} \ \ \forall \ k, g \in G^k \end{array}$$

- Branching on a aggregate original var.: $\boldsymbol{y_i = \sum_{g \in G} x_i^g \nu_g}$ is **not an option** as $y_i = 1$ in all master LP solutions.
- Branching on a pair of original var. or equivalently on an **auxiliary variable** $\boldsymbol{w_{ij}}$, using

$$(w_{ij} = \sum_{g: x_i^g = 1, x_j^g = 1} \nu_g = 0) \quad \text{or} \quad (w_{ij} = \sum_{g: x_i^g = 1, x_j^g = 1} \nu_g = 1)$$

# Branching

## Branching Options : The set partitioning case

Up-branch $\quad (w_{ij} = \sum_{g: x_i^g = 1, x_j^g = 1} \nu_g = 1)$

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- Option 3: Branching constraint in the master **[Des09]**

$$\sum_{g:x_i^g=1,x_j^g=1} \nu_g \geq 1 \quad (\mu)$$

$$\zeta_3 = \min\{(c-\pi D)x - \mu w_{ij} : x \in Z, w_{ij} \leq x_i, w_{ij} \leq x_j, w_{ij} \in \{0,1\}\}.$$

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- Option 5: Differentiate 2 pricing problems, and enforce Branch. Constr. explicitly in the sub-prob: **[Van09]**

$$\zeta_{5A} = \min\{(c - \pi D)x : x \in Z, x_i = x_j = 0\}$$

$$\zeta_{5B} = \min\{(c - \pi D)x : x \in Z, x_i = x_j = 1\}.$$

$\sum_{g \in G_{5A}} \nu_g = K - 1$ and $\sum_{g \in G_{5B}} \nu_g = 1$

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Strength of the LP bound, Complexity of the SP, Interior Points

# Branching

## Multiple identical subsystems: the general case

- Option 1: Branch on the aggregate variables $y_i = \sum_{g \in G} x_i^g \nu_g^* = \alpha \notin \mathbb{Z}$ in the master

$$\boldsymbol{\sum_{g \in G} x_i^g \nu_g \leq \lfloor \alpha \rfloor \ \text{ or } \ \sum_{g \in G} x_i^g \nu_g \geq \lceil \alpha \rceil.}$$

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- Option 3 & 4: Branch on auxiliary variables (implicit reformulation) in the Master or the SubProb..

  (SDVRP Example of opt 3: $z_{ijk} = x_{ij} x_{jk}$ **[Des09]**)

## Multiple identical subsystems: the general case

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$$\sum_{g \in G} \boldsymbol{x_i^g \nu_g} \leq \lfloor \boldsymbol{\alpha} \rfloor \ \textbf{or} \ \sum_{g \in G} \boldsymbol{x_i^g \nu_g} \geq \lceil \boldsymbol{\alpha} \rceil.$$

- Option 3 & 4: Branch on auxiliary variables (implicit reformulation) in the Master or the SubProb..
  (SDVRP Example of opt 3: $z_{ijk} = x_{ij} x_{jk}$ **[Des09]**)
- Option 2 & 5: Branch on one (or several) components of $x$ and differentiate subproblems: if $\sum_{g : x_j^g \geq l_j} \nu_g^* = \alpha \notin \mathbb{Z}$,

$$\sum_{g : x_j^g \geq l_j} \boldsymbol{\nu_g} \geq \lceil \boldsymbol{\alpha} \rceil \ \textbf{or} \ \sum_{g : x_j^g \leq l_j - 1} \boldsymbol{\nu_g} \geq \boldsymbol{K} - \lfloor \boldsymbol{\alpha} \rfloor$$

  Pricing is carried out independently over each SP set $\hat{Z} = Z \cap \{x_j \geq l_j\}$ and $Z \setminus \hat{Z}$ in both branches. **[Van09]**

# Aggregate integrality property

$$
\begin{array}{rcl}
\min \sum_k cx^k & & \\
\sum_k Dx^k & \geq & d \\
Bx^k & \geq & b \ \forall k \\
x^k & \in & \mathbb{N}^n \ \forall k \\
\multicolumn{3}{c}{Z^k := \{x^k \in \mathbb{N}^n : \ Bx^k \geq b\}}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{rcl}
\min cy & & \\
y & = & \sum_k x^k \\
Dy & \geq & d \\
x^k & \in & \mathit{conv}(Z^k) \ \forall k \\
y & \in & \mathbb{N}^n
\end{array}
$$

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Dy & \geq & d \\
x^k & \in & \mathit{conv}(Z^k) \ \forall k \\
y & \in & \mathbb{N}^n
\end{array}
$$

## Proposition 1

The Bin Packing Problem formulated using arc flow variables has the *"Aggregate integrality property"*.

# Aggregate integrality property

$$\begin{array}{rcl} \min \sum_k cx^k & & \\ \sum_k Dx^k & \geq & d \\ Bx^k & \geq & b \; \forall k \\ x^k & \in & \mathbb{N}^n \; \forall k \\ Z^k := \{x^k \in \mathbb{N}^n : & Bx^k \geq b\} & \end{array} \quad \Leftrightarrow \quad \begin{array}{rcl} \min cy & & \\ y & = & \sum_k x^k \\ Dy & \geq & d \\ x^k & \in & conv(Z^k) \; \forall k \\ y & \in & \mathbb{N}^n \end{array}$$

## Proposition 1

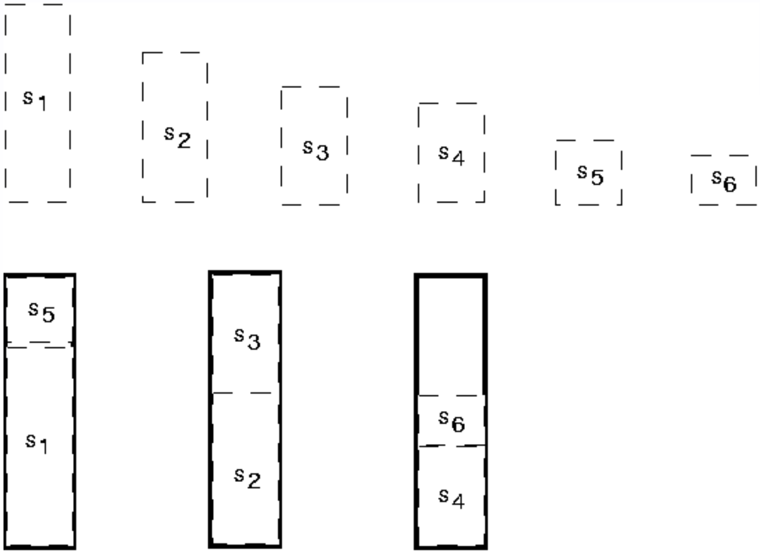
The Bin Packing Problem formulated using arc flow variables has the *"Aggregate integrality property"*.

## Proposition 2

The Vehicle Routing Problem formulated using edge flow variables has the *"Aggregate integrality property"*.

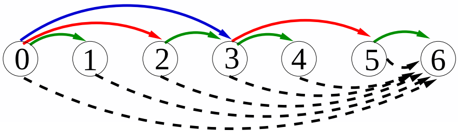
# Aggregate integrality property

## The Bin Packing Problem



## Bin Packing: Network Flow formulation **[Val99]**

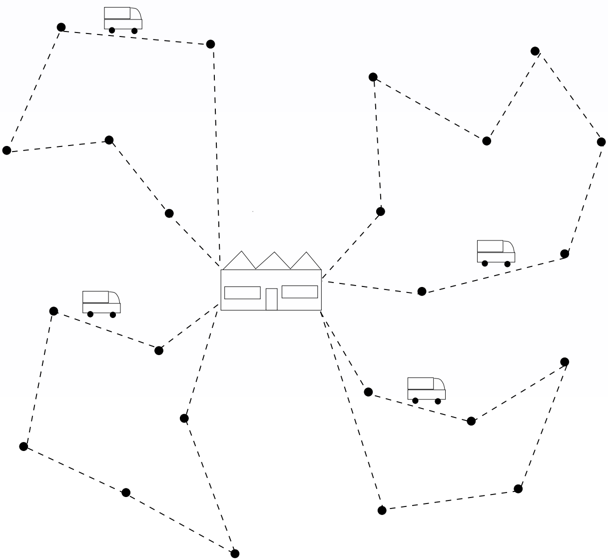
$$
\begin{array}{rcl}
\min \sum_{i,v} F^i_{ov} & & \\
\sum_{(u,v)} F^i_{uv} & = & 1 \quad \forall i \\
F^i_{uv} & = & \sum_k f^{ik}_{uv} \quad \forall (u,v) \\
f^k & \in & \mathit{conv}(Z^k) \quad \forall k \\
F^i_{uv} & \in & \{0,1\} \quad \forall i, (u,v) : v = u + s_i
\end{array}
$$




$$
\mathbf{Z^k} = \{ f \in \{0,1\}^{n*m} \; : \; \begin{array}{l}
\sum_{i,u} f^i_{uv} = \sum_{i,u} f^i_{vu} + f_{v,C} \quad v = 1,.,C-1 \\
\sum_{i,u} f^i_{uC} + \sum_v f_{v,C} = 1 \\
0 \le f^i_{uv} \le 1 \; \forall i, u, v = u + s_i \}
\end{array}
$$

# Aggregate integrality property

## The Vehicle Routing Problem



# Aggregate integrality property

## The Vehicle Routing Problem

$$\begin{aligned}
\min \sum_{e \in E} c_e \; x_e & & \\
\sum_{e \in \delta(i)} x_e &= 2 \quad \forall i \in V \setminus \{0, n+1\} \\
\sum_{e \in \delta(i)} x_e &= K \quad \forall i \in \{0, n+1\} \\
x_e &= \sum_k (z_{ij}^k + z_{ji}^k) \quad \forall e = (i,j) \\
z^k &\in \textit{conv}(Z^k) \quad \forall k \\
x_e &\in \{0,1\} \quad \forall e \in E,
\end{aligned}$$

$$\begin{aligned}
\mathbf{Z^k} \quad = \quad & \{(y,z) \in \{0,1\}^{n+m} : \sum_j z_{ji} = \sum_j z_{ij} = y_i \forall i \\
& \sum_i d_i y_i \leq C, \sum_{(ij) \in \delta(S)} z_{ij} \geq y_v \forall v, S \subset V \setminus \{0, n+1\} : S \ni v\}
\end{aligned}$$

# Extended original formulation

## Bin Packing: reformulation as a VRP **[Bel05]**

$$
\begin{aligned}
\min \quad & \sum_{i \in V} x_{0i} \\
\sum_{(i,j)} x_{ij} = \sum_{(j,i)} x_{ji} \quad & = \quad 1 \qquad\qquad \forall i \\
\sum_{(ij) \in \delta(S)} x_{ij} \quad & \geq \quad \frac{d(S)}{C} \qquad\qquad \forall S \subseteq V \\
x_{ij} \quad & \in \quad \{0,1\} \qquad \forall i,j
\end{aligned}
$$

# Extended original formulation

## Bin Packing: reformulation as a VRP **[Bel05]**

$$\min \sum_{i \in V} x_{0i}$$

$$\sum_{(i,j)} x_{ij} = \sum_{(j,i)} x_{ji} \quad = \quad 1 \qquad \forall i$$

$$\sum_{(ij) \in \delta(S)} x_{ij} \quad \geq \quad \frac{d(S)}{C} \qquad \forall S \subseteq V$$

$$x_{ij} \quad \in \quad \{0,1\} \qquad \forall i,j$$

- **Symmetry breaking:** $x_{ij}$ only defined for $i < j$.
- **Specific oracle:** resource constrained shortest path.
- *"Aggregate integrality property"*.

# Classification

1] Branching on

- a var of the Dantzig-Wolfe reformulation, (for heuristics)
- a var of the original formulation, [Vil05]
- extended original formul. var. [Bel05, Val99, Des09]
- a set of var of the original form. [RF81, Ban00, Van00, Van09]

2] Branching constraints are added

- to the master **(soft branching)**, [Bel05, Val99, Van00, Des09]
- to the subproblem **(hard branching)**. [RF81, Ban00, Vil05, Van09]

3] Branching constraints

- yield subproblem modifications, [RF81, Van00, Des09]
- preserve **subproblem structure**. [Bel05, Val99, Ban00, Vil05, Van09]

4] The branching scheme

- enumerates symmetric solutions, [Vil05, Val99, Bel05]
- avoids the **symmetry drawback**. [RF81, Ban00, Van00, Des09, Van09]

5] The branching scheme is

- application / oracle **specific**, [Bel05], [Val99], [Ban00], [Des09]
- **generic**. [RF81], [Vil05], [Van00], [Van09]

# Numerical Comparisons

## Bin Packing: Facility Location Formulation

$$\begin{aligned} \min \quad & \sum_{k=1}^{n} x_{kk} & & \\ \sum_{k=1}^{n} x_{ki} & = 1 & & \forall i \\ \sum_{i=1}^{n} s_i \, x_{ki} & \leq W x_{kk} & & \forall k \\ x_{ki} & \in \{0,1\} & & \forall i,k : k < i \end{aligned}$$

**Symmetry breaking:**

$x_{kk} = 1$ if facility $k$ is open & item $k$ is the smallest index on it.

$x_{ki} = 1$ if item $i$ is assigned to facility $k$ on the same bin as $k$

**Multiple Non-identical Sub-systems**

# Numerical Comparisons

average over 10 random BPP instances, oracle = Cplex

$n = 100, \ w_i \sim U[500, 2500], \ W = 10000$

| Facility Location (Option 1) | | | | | | | |
|---|---|---|---|---|---|---|---|
| nodes | depth | #mast | #sp | #col | Tmast | Tsp | time |
| 68 | 67 | 1449 | 6120 | 5584 | 1557 | 2998 | 1m59 |
| **Network Flow (Option 1) ($n = 50$ & $100$, $W = 1000$)** | | | | | | | |
| nodes | depth | #mast | #sp | #col | Tmast | Tsp | time |
| 44 | 43 | 2248 | 1760 | 1760 | 2023 | 52511 | 19m58s |
| >3k | >129 | 124k | 94k | 76k | 1.5kk | 5kk | >71h |
| **Ryan-Foster (Option 4)** | | | | | | | |
| nodes | depth | #mast | #sp | #col | Tmast | Tsp | time |
| 75 | 74 | 2609 | 666 | 666 | 556 | 544 | 0m17s |
| **Component Bound Set (Option 5)** | | | | | | | |
| nodes | depth | #mast | #sp | #col | Tmast | Tsp | time |
| 21 | 20 | 4124 | 2880 | 2880 | 3355 | 2631 | 1m49s |

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