ON THE SOLVABILITY OF FOURTH-ORDER BOUNDARY VALUE PROBLEMS WITH ACCRETIVE OPERATORS

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Abstract. In this paper, we consider an abstract fourth order boundary value problem where the coefficients are accretive operators in Hilbert space. We show existence, uniqueness and maximal regularity of the solution under some necessary and sufficient conditions on the data. To this end, we give an explicit representation formula, using analytic semigroups, sectorial operators with Bounded Imaginary Powers, the theory of strongly continuous cosine operator functions and the perturbation theory of m-accretive operators. Illustrative example is also given.

1. introduction

The main purpose of this paper is to investigate the solutions of the following fourth order abstract linear differential equation under several sets of assumptions

$$u^{(4)}(x) - 2Bu''(x) - Cu(x) = f(x), \qquad x \in (a, b), \tag{1.1}$$

where $u(x)$ is a vector-valued function in an appropriate (finite or infinite dimensional) Hilbert space $\mathcal{H}$, B and C are linear (bounded or unbounded) accretive operators on $\mathcal{H}$, while the function $f \in L^p(a, b, \mathcal{H})$, $1 < p < +\infty$, $a < b$. Here we are concerned with the existence, uniqueness and L^p-maximal regularity of the solutions of (1.1) when this problem is supplemented with a suitable types of nonhomogeneous boundary conditions.

By using splitting method of Krein, [15], the equation is transformed to the study of two coupled equations of second order. The first one is an elliptic problem and the second one is an hyperbolic problem. They are not directly accessible by standard methods. The main difficulty is to analyse this two kinds of problems simultaneously.

Several authors have studied equation of second order when it is regarded as an abstract problem of parabolic or hyperbolic type. See, for instance [6], [7], [8], [15], [24], [30] and references therein cited. Fourth-order equations as (1.1) also arise in a variety of physical problems as in [16], [17] and [29].

We extend the work done in [29], where a biharmonic equation has been formulated as (1.1) with $C = -B^2$. But by the same splitting method this work leads to the study of two coupled elliptic equations of second order. An explicit representation formula of the solution to (1.1) was given under various boundary conditions. This was done with the help of the operator $\sqrt{B}$ as a root of the characteristic operational equation, $X^4 - 2BX + B^2X = 0$ for what the operator B was assumed to be sectorial operator,

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then the existence of $\sqrt{B}$ was ensured, see [29]. The representation formula of the solution is based on the operator $-\sqrt{B}$ which generates an analytic semi-group. The analogy of this idea to the general case, up to now is still not obvious.

In our analysis, the coupled hyperbolic and elliptic equations of second order are solved to give an explicit solution to the problem (1.1) with suitable boundary conditions. Our approach is essentially based on the fractional powers of m-accreive operators, the perturbation theory of such operators, the techniques of the holomorphic semigroups generated by them and also the theory of strongly continuous cosine operator functions.

Remarkably, for the first time, the solution given is expressed in term of analytic semigroups and strongly continuous cosine operator functions. Such results are interesting, firstly because they provide an existence, uniqueness and L^p-maximal regularity of the solution for the fourth order abstract boundary value problem (1.1) under some necessary and sufficient conditions on the data. Moreover, they are of interest regarded as an application of the perturbation theory of m-accretive operators, [10], [11], [13], [20], [21], [22], [32]. This allows us to find various sufficient conditions on the accretive operators B and C, under which our results remains valid.

Our paper is organised as follows. Before to give our mains results, we review in Section 2 some basic concepts, notation and properties needed. The precise formulations of the results of the present work will be given in Section 3, where we describe the problem and the main assumptions. We also give technical results which help us to prove our main result. In Section 4, an example of initial boundary problem, to which theory applies, is given.

2. Preliminaries

Throughout this paper $\mathcal{H}$ is a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For a closed linear operator A on $\mathcal{H}$ we denote by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$ and $\rho(A)$ the domain, the range, the kernel, the spectrum and the resolvent set of A, respectively. The space of bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For two possibly unbounded linear operators A, B on $\mathcal{H}$ their product AB is defined on its natural domain $\mathcal{D}(AB) := \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$ and their sum $A + B$ is defined in $\mathcal{D}(A+B) = \mathcal{D}(A)\cap\mathcal{D}(B)$. An inclusion $A \subseteq B$ denotes inclusion of graphs, *i.e.*, it means that B extends A. A possibly unbounded operator A on $\mathcal{H}$ commutes with a bounded operator $C \in \mathcal{B}(\mathcal{H})$ if the graph of A is $C \times C$-invariant, or equivalently if $CA \subseteq AC$.

2.1. **Accretive operators framework.**

The numerical range of an operator $A : \mathcal{D}(A) \to \mathcal{H}$ is the set

$$W(A) := \{\langle Ax, x\rangle : \quad x \in \mathcal{D}(A), \quad \|x\| = 1\},$$

It is known that the set $W(A)$ is convex set of the complex plane (the Toeplitz-Hausdorff theorem), and in general is neither open nor closed, even for a closed operator A. Furthermore, if A is a bounded operator then $W(A)$ has the so-called spectral inclusion property

$$\sigma(A) \subset \overline{W(A)}. \tag{2.1}$$

For unbounded operator additional assumptions are needed to ensure (2.1), this is the case, when $\mathbb{C} \setminus \overline{W(A)}$ is connected and contains a point of $\rho(A)$, see Theorem V.3.2 in

[13]. In particular, in the maximal accretive operators case it is always satisfied. We now recall the definition of accretive operators.

Definition 2.1. Let A be an operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$. We say that A is *accretive* if

$$W(A) \subseteq \overline{\mathbb{C}_+} = \{\lambda \in \mathbb{C} : \quad \mathrm{Re}(\lambda) \geq 0\}.$$

If A is accretive and $(I+A)$ is surjective, we say that A is *maximal accretive*, or *m-accretive* for short.

Operator A is called maximal accretive if for every accretive operator S with $A \subset S$ it follows that $A = S$. Then A is m-accretive if and only if A is accretive and has no proper accretive extensions in $\mathcal{H}$.

In particular, every m-accretive operator is accretive, closed and densely defined (see [13], p. 279). Furthermore,

$$(A + \lambda I)^{-1} \in \mathcal{B}\ (\mathcal{H}) \quad \text{ and } \quad \left\|(A + \lambda I)^{-1}\right\| \leq \frac{1}{\lambda}, \quad \text{for } \lambda > 0.$$

In particular, a bounded accretive operator is m-accretive.

An operator A is called dissipative (resp. m-dissipative) if $-A$ is accretive (resp. m-accretive). A normal operator A (bounded or not) is m-accretive if and only if its spectrum is contained in the half complex plane $\overline{\mathbb{C}_+}$. Hence a normal accretive operator is m-accretive. Next we collect some basic results about linear accretive operators, see for instance [13] or [14].

Lemma 2.2. *Let A be an accretive operator. Then $\mathcal{R}(I + A)$ is closed if and only if A is closed.*

Proposition 2.3. *Let A be an m-accretive operator. Then we have the following.*
(1) *A^* is m-accretive.*
(2) *$\mathcal{N}(A) = \mathcal{N}(A^*)$.*
(3) *If A is injective, then $A \in BIP(\mathcal{H}, \pi/2)$.*

Here, $BIP(\mathcal{H}, \theta)$ denotes the class of Bounded Imaginary Powers operators of angle θ on $\mathcal{H}$.

For more details on BIP operators, we refer to [25] or [27].

Proposition 2.4. *Let A be a densely defined closed accretive operator. Then A is m-accretive if and only if A^* is accretive.*

For $\omega \in [0, \pi/2)$, we denote by

$$\mathcal{S}(\omega) := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \omega\},$$

the open sector of $\mathbb{C} \setminus \{0\}$ with semi-angle ω. An operator A will be said is *ω-accretive* if

$$W(A) \subset \overline{\mathcal{S}(\omega)} := \{z \in \mathbb{C} : |\arg(z)| \leq \omega\},$$

or, equivalently,

$$|\mathrm{Im}\,(\langle Ax, x\rangle)\,| \leq \tan(\omega)\,\mathrm{Re}\,(\langle Ax, x\rangle)\,, \quad \text{for all } x \in \mathcal{D}(A).$$

An ω-accretive operator A is called m-ω-accretive, if it is m-accretive. We have A is m-ω-accretive if and only if operators $e^{\pm i\theta}A$ are m-accretive for $\theta = \frac{\pi}{2} - \omega$, $0 < \omega \leq \pi/2$.

If $\omega = \pi/2$, then we have $\overline{\mathcal{S}(\pi/2)} = \overline{\mathbb{C}_+}$ and m-$\pi/2$-accretivity means m-accretivity. Also, m-0-accretive is the nonegative selfadjoint operator.

The resolvent set of an m-ω-accretive operator A contains the set $\mathbb{C} \setminus \overline{\mathcal{S}(\omega)}$ and

$$\|(A - \lambda I)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \mathcal{S}(\omega))}, \quad \lambda \in \mathbb{C} \setminus \overline{\mathcal{S}(\omega)}.$$

Consequently, the numerical range of m-ω-accretive operator, $0 \leq \omega \leq \pi/2$, has the so-called spectral inclusion property (2.1).

It is well known that if A is m-ω-accretive, $0 \leq \omega < \pi/2$, then it generates contractive C_0-semigroup $\mathcal{T}(t) = e^{-tA}$, $t \geq 0$, and has an holomorphic continuation into the sector $\overline{\mathcal{S}(\pi/2 - \omega)}$, see Theorem IX-1.24 in [13]. In this case, we can write $\mathcal{T}(z) = e^{-zA}$, for all $z \in \overline{\mathcal{S}(\pi/2 - \omega)}$; moreover, we have

$$\left\|e^{-zA}\right\| \leq 1, \quad z \in \overline{\mathcal{S}(\pi/2 - \omega)}.$$

See also [3], for more details.

2.2. **Interpolation spaces.**

Recall that the general definition of the real interpolation space $(X_0, X_1)_{\theta,q}$ with X_0, X_1 two Banach spaces such that $X_0 \hookrightarrow X_1$, is described for instance in [19] or in [31]. For the reader convenience, we give here a definition adapted to our case.

Definition 2.5. Let X be a Banach space, and $A : \mathcal{D}(A) \subset X \longrightarrow X$ a linear operator such that

$$(0, +\infty) \subset \rho(A) \quad \text{and} \quad \exists\, C > 0 : \forall\, t > 0, \quad \|t(A - tI)^{-1}\|_{\mathcal{B}(X)} \leqslant C. \tag{2.2}$$

Then, from [9], Teorema 3, p. 665, for $\theta \in (0, 1)$ and $q \in [1, +\infty)$, we can define the real interpolation space

$$(\mathcal{D}(A), X)_{\theta,q} = \left\{\psi \in X : \left(\int_0^{+\infty} t^{1-\theta} \|A(A - tI)^{-1}\psi\|_X^q \frac{dt}{t}\right)^{1/q} < +\infty\right\}.$$

Note that in [31], this space is denoted by $\big(X, \mathcal{D}(A)\big)_{1-\theta,q}$.

2.3. **Strongly continuous cosine family.**

We recall the definition of a strongly continuous cosine family. For more information we refer to [1] or [5].

Definition 2.6. A family $(Cos(t))_{t\in\mathbb{R}}$ of bounded linear operators on $\mathcal{H}$ is called a *cosine family* when the following two conditions hold

(1) $Cos(0) = I$.
(2) For all $t, s \in \mathbb{R}$, we have

$$2Cos(t)Cos(s) = Cos(t + s) + Cos(t - s). \tag{2.3}$$

It is defined to be *strongly continuous*, when for all $x \in \mathcal{H}$ and all $t \in \mathbb{R}$, we have

$$\lim_{h\to 0} Cos(t + h)x = Cos(t)x.$$

Similarly than for strongly continuous semigroups we can define the infinitesimal generator.

Definition 2.7. Let $(Cos(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family, then the *infinitesimal generator* A is defined as

$$Ax = \lim_{t\to 0} \frac{2(Cos(t)x - x)}{t^2},$$

with its domain

$$\mathcal{D}(A) = \left\{ x \in \mathcal{H} : \quad \lim_{t\to 0} \frac{2(Cos(t)x - x)}{t^2} \text{ exists} \right\}.$$

This infinitesimal generator is a closed densely defined operator. The following well-known estimates are needed. For a proof we refer to section 3. of [5].

Lemma 2.8. *Let A be a closed densely defined operator in $\mathcal{H}$. Then A is the generator of a strongly continuous cosine family $Cos(t)_{t\in\mathbb{R}}$ satisfying*

$$\|Cos(t)\| \leq me^{r|t|}, \quad \text{for all } t \in \mathbb{R}, \tag{2.4}$$

for some $r \geq 0$ and $m > 0$ if and only if, for all $\lambda > r$, we have

$$\left\{\lambda^2 : \mathrm{Re}(\lambda) > r\right\} \subset \rho\left(A\right),$$

and for every $n \in \mathbb{N}$

$$\left\| \frac{d^n}{d\lambda^n} \left[\lambda \left(\lambda^2 - A\right)^{-1} \right] \right\| \leq \frac{n!\, m}{(\lambda - r)^{n+1}}.$$

Hence the above lemma shows that the spectrum of A must lie within the parabola $\{z \in \mathbb{C} \mid z = \lambda^2 \text{ with } \mathrm{Re}(\lambda) = r\}$.

It is known that A is closed and densely defined. The operator family $(S(t))_{t\in\mathbb{R}}$ given by

$$Sin(t) = \int_0^t Cos(s)ds, \quad \text{for all } t \in \mathbb{R},$$

is called the associated sine family.

3. A fourth order linear boundary value problem

The aim of this section is to give various kind of assumptions which allow us to determine a representation formula of the solution to the following equation

$$u^{(4)}(x) - 2Bu''(x) - Cu(x) = f(x), \quad x \in (a,b), \tag{3.1}$$

where $f \in L^p(a,b;\mathcal{H})$, $\mathcal{H}$ is a Hilbert space, B and C are two operators with domain $\mathcal{D}(B)$ and $\mathcal{D}(C)$, respectively.

3.1. **Assumptions and statement of main results.**

Assume that

(H_1) B^2 and C are accretive operators, with $\mathcal{D}(B^2) \subset \mathcal{D}(C)$,

(H_2) $0 \in \rho(B^2 + C)$,

(H_3) $\mathcal{D}(\sqrt{B^2 + C}) \subset \mathcal{D}(B)$,

(H_4) $B\sqrt{B^2 + C} = \sqrt{B^2 + C}\, B$,

(H_5) $0 \in \rho(B + \sqrt{B^2 + C})$,

(H_6) There exists $\nu > 0$ such that
$$W(\sqrt{B^2 + C} - B) \subset \{z \in \mathbb{C} : \quad 4\nu^2 \mathrm{Re}(z) \geqslant \mathrm{Im}(z)^2\}.$$

In all the sequel, we set

$$L = B + \sqrt{B^2 + C} \quad \text{and} \quad M = B - \sqrt{B^2 + C}.$$

The existence of L and M is ensured by Lemma 3.7 below. Thus, equation (3.1) reads as

$$u^{(4)}(x) - (L + M)u''(x) + LMu(x) = f(x), \quad x \in (a, b). \tag{3.2}$$

We call classical solution of (3.2), a solution of (3.2) such that

$$u(\cdot) \in W^{4,p}(a, b; H) \cap L^p(a, b; D(LM)) \quad \text{and} \quad u''(\cdot) \in L^p(a, b; D(L + M)). \tag{3.3}$$

In the sequel, we will say that a classical solution to a boundary problem is a classical solution to the equation of the problem satisfying the boundary conditions.

Theorem 3.1. *Let $f \in L^p(a, b; \mathcal{H})$ with $a, b \in \mathbb{R}$, $a < b$ and $p \in (1, +\infty)$. Assume that (H_1), (H_2), (H_3), (H_4), (H_5) and (H_6) hold. Then, there exists a unique classical solution F_U, where $U = (u_1, u_2, u_3, u_4)$, to problem*

$$\left\{\begin{array}{lllll} u^{(4)}(x) - (L + M)u''(x) + LMu(x) = f(x), \quad x \in (a, b), & & & & \\ u(a) & = & u_1, & u'(a) & = u_2 \\ u''(a) & = & u_3, & u''(b) - Mu(b) & = u_4, \end{array}\right. \tag{3.4}$$

if and only if

$$\left\{\begin{array}{l} u_1, u_2, u(b) \in \mathcal{D}(L), \\ Lu_2 \in \left(\mathcal{D}\left(\sqrt{L}\right), \mathcal{H}\right)_{\frac{1}{p},p}, \\ Lu_1, Lu(b), u_3, u_4 \in (\mathcal{D}(L), \mathcal{H})_{\frac{1}{2p},p}. \end{array}\right. \tag{3.5}$$

Moreover, F_U is given by

$$F_U(x) := Cos(x - a)u_1 + Sin(x - a)u_2 + \int_a^x Sin(x - s)v(s)ds, \tag{3.6}$$

where

$$\begin{aligned} v(x) &:= \left(e^{-(x-a)\sqrt{L}} - e^{-(b-x)\sqrt{L}}e^{-c\sqrt{L}}\right) W\left(u_3 - Mu_1\right) \\ &+ \left(e^{-(b-x)\sqrt{L}} - e^{-(x-a)\sqrt{L}}e^{-c\sqrt{L}}\right) Wu_4 \\ &+\frac{1}{2}\left(e^{-(b-x)\sqrt{L}}e^{-c\sqrt{L}} - e^{-(x-a)\sqrt{L}}\right) W\sqrt{L}^{-1}\int_a^b e^{-(s-a)\sqrt{L}}f(s)\,ds \\ &+\frac{1}{2}\left(e^{-(x-a)\sqrt{L}}e^{-c\sqrt{L}} - e^{-(b-x)\sqrt{L}}\right) W\sqrt{L}^{-1}\int_a^b e^{-(b-s)\sqrt{L}}f(s)\,ds \\ &+\frac{1}{2}\sqrt{L}^{-1}\int_a^x e^{-(x-s)\sqrt{L}}f(s)\,ds + \frac{1}{2}\sqrt{L}^{-1}\int_x^b e^{-(s-x)\sqrt{L}}f(s)\,ds, \end{aligned} \tag{3.7}$$

with $c := b-a$, $W := \left(I - e^{-2c\sqrt{L}}\right)^{-1}$, $Cos(\cdot)$ *is the cosine function family generated by* $-M$ *and* $Sin(\cdot)$ *is the associated sine function.*

As a consequence of the previous result and [29], Lemma 3.1, p. 638, we obtain the following result.

Corollary 3.2. *Let* $f \in L^p(a,b;\mathcal{H})$ *with* $a,b \in \mathbb{R}$, $a<b$ *and* $p \in (1,+\infty)$. *Assume that* (H_1), (H_2), (H_3), (H_4), (H_5) *and* (H_6) *hold. Then, there exists a unique classical solution to problem*

$$\begin{cases} u^{(4)}(x) - (L+M)u''(x) + LMu(x) = f(x), \quad x \in (a,b) \\ u(a) = u_1, \quad u'(a) = u_2 \\ u''(a) = u_3, \quad u''(b) = u_4, \end{cases}$$

if and only if (3.5) *holds. This solution is given by* $F_{(u_1,u_2,u_3,u_4+Mu(b))}$.

Under other boundary conditions, equation (3.1) is more complicated to study. However, the following result state a representation formula for the classical solution of equation (3.1).

Theorem 3.3. *Let* $f \in L^p(a,b;\mathcal{H})$ *with* $a,b \in \mathbb{R}$, $a<b$ *and* $p \in (1,+\infty)$. *Assume that* (H_1), (H_2), (H_3), (H_4), (H_5) *and* (H_6) *hold. If* u *is a classical solution of* (3.1), *then there exist* K_1, K_2, K_3, $K_4 \in \mathcal{H}$ *such that*

$$\begin{aligned} u(x) &= Cos(x-a)K_1 + Sin(x-a)K_2 \\ &\quad + e^{(x-a)\sqrt{L}}K_3 + e^{(b-x)\sqrt{L}}K_4 + F_0(x), \end{aligned} \tag{3.8}$$

where F_0 *corresponds to* F_U *defined in Theorem 3.1 with* $U = (0,0,0,0)$.

Now, we give some new assumptions to replace (H_1) and (H_2). Under these new assumptions, we state that the previous results remains true.

Theorem 3.4. *Let* B^2 *is m-accretive and* $C - \gamma I$ *is accretive for some* $\gamma > 0$ *such that* $\mathcal{D}(B^2) \subset \mathcal{D}(C)$. *Theorem 3.1 and Theorem 3.3 remains valid if we replace* (H_1) *and* (H_2) *by one of the following assumptions:*

(A.1) *There are nonnegative constants* $\alpha \geq 0$, $0 \leq \beta < 1$ *and* $\delta \geq 0$ *such that*

$$\operatorname{Re}\left(\left\langle B^2u, Cu\right\rangle\right) \geq -\alpha\left\|u\right\|^2 - \beta\left\|B^2u\right\|^2 - \delta\left\|B^2u\right\|\left\|u\right\|, \tag{3.9}$$

for all $u \in \mathcal{D}(B^2)$.

(**A.2**) *C is B^2-bounded with lower bound lesser than 1. This means that there exist $\alpha \geq 0$ and $0 \leq \beta < 1$ such that*

$$\|Cu\|^2 \leq \alpha \|u\|^2 + \beta \left\|B^2 u\right\|^2, \quad \text{for all } u \in \mathcal{D}(B^2). \tag{3.10}$$

(**A.3**) *$I + (C - \gamma I)(B^2 + t_0 I)^{-1}$ is boundedly invertible, for some $t_0 > 0$.*
(**A.4**) $\sup_{t>0} \left\|C(B^2 + tI)^{-1}\right\| < 1$.
(**A.5**) *B is an accretive operator with $\mathcal{D}(B) \subset \mathcal{D}(C)$.*
(**A.6**) *$\mathcal{D}((B^2)^\alpha) \subset \mathcal{D}(C)$, for some $0 < \alpha < 1$.*
(**A.7**) *Operator C is bounded.*

In general, statement of Theorem 3.4 is not true if we only assume that B is accretive. In fact $B^2 + C$ not need to be m-accretive, because B^2 fails to be accretive (with the same vertex) even in the case of an accretive matrix B with numerical range contained in a sector of angle lesser than $\pi/4$, see Example 1.2 in [11].

Conversely, if B^2 is m-accretive; then B fails to be accretive. For instance if $B = i\dfrac{d}{dx}$ on $L^2(\mathbb{R})$, its spectrum is on both sides of the origin. But $B^2 = -\dfrac{d^2}{dx^2}$ is a nonegative selfadjoint operator. Also the existing general numerical range mapping theorems do not encompass the question of when B accretive implies B^2 accretive. In [11] the authors considered this question, for both bounded and unbounded sectorial operators. Now, the accretivity of B^2 is relaxed and replaced by some conditions discussed in [11] to guarantees (H_1) and (H_2).

Theorem 3.5. *Let $C - \gamma I$ be m-ω_1-accretive operator for some $\gamma > 0$ with $0 \leq \omega_1 \leq \pi/2$. Assume that C and B^2 verify one of the assumptions (**A.1**)-(**A.6**) with interchanging the role of B^2 and C. If B is an ϑ-accretive operator in $\mathcal{H}$ with $\vartheta < \pi/2$ such that $\mathcal{D}(B) \subset \mathcal{D}(\mathrm{Re(B)}) \cap \mathcal{D}(\mathrm{Im(B)})$ and*

$$\|\mathrm{Im}(B)u\| \leq c \|\mathrm{Re}(B)u\|, \quad \text{for all } u \in \mathcal{D}(B^2), \tag{3.11}$$

for some $c \leq 1$. Then $B^2 + C - \gamma I$ is m-ω-accretive for some $\gamma > 0$ with $\omega = \max\{\omega_1, \omega_2\}$ where $\omega_2 \leq 2\arctan(c)$.

The following result represents the bounded case (**A.7**) which follows as a consequence of [11], Corollary 4.2, p. 700.

Corollary 3.6. *Let $C - \gamma I$ be a bounded ω_1-accretive operator for some $\gamma > 0$ with $0 \leq \omega_1 \leq \pi/2$ and B is bounded operator such that*

$$\|\mathrm{Im}(B)u\| \leq c \|\mathrm{Re}(B)u\|, \quad \text{for all } u \in \mathcal{H}, \tag{3.12}$$

and for some $c \leq 1$. Then $B^2 + C - \gamma I$ is bounded ω-accretive for some $\gamma > 0$ with $\omega = \max\{\omega_1, \omega_2\}$ where $\omega_2 \leq 2\arctan(c)$ and $\arctan(c) \in [0, \pi/2]$.

3.2. **Prerequisites.**

In this section, we recall and state some technical results which will allow us to prove the previous theorems.

Lemma 3.7. *Under (H_1) and (H_2), operator $B^2 + C$ with domain $\mathcal{D}(B^2)$ is m-accretive. In particular, the square root $\sqrt{B^2 + C}$ is well-defined and unique.*

Moreover $\sqrt{B^2+C}$ *is a m-*$(\pi/4)$*-accretive operator and* $\mathcal{D}(B^2)$ *is a core of* $\sqrt{B^2+C}$*. This means that the closure of the restriction of* $\sqrt{B^2+C}$ *to* $\mathcal{D}(B^2)$ *is again* $\sqrt{B^2+C}$*.*

Proof. Due to (H_1), B^2+C with domain $\mathcal{D}(B^2)$ is an accretive operator.

Moreover, (H_2) implies that B^2+C+tI is invertible with bounded inverse, for $0 \leq t < \varepsilon$, for some $\varepsilon > 0$. By Proposition 3.14.(ii) of [4] p. 82, we conclude that B^2+C+tI is invertible with bounded inverse for all $t > 0$. Thus B^2+C is m-accretive. The rest of the lemma is an immediate consequence of [13], Theorem 3.35, p. 281. □

Lemma 3.8. *Under* (H_1)*,* (H_2) *and* (H_3)*, we have*

(1) *for any* $\varepsilon > 0$*,*

$$L \text{ and } -M \text{ are m-}\psi\text{-accretive operators with } \psi = \pi/4 + \varepsilon.$$

In particular, $-L$ *and* M *generates holomorphic* C_0*-semigroup of contraction.*

(2) $\sqrt{L}$ *and* $\sqrt{-M}$ *are m-*ϕ*-accretive with* $\phi = \psi/2$*.*

(3) $L, -M \in BIP(\mathcal{H}, 3\pi/4)$*.*

Proof.

(1) Since $\mathcal{D}(\sqrt{B^2+C}) \subset \mathcal{D}(B)$ and $\mathcal{D}(\sqrt{B^2+C})$ is dense in $\mathcal{H}$, so is $\mathcal{D}(B)$. Since $\sqrt{B^2+C}$ is m-$(\pi/4)$-accretive and invertible, there exists $c_1 > 0$ such that

$$\|Bu\| \leq c_1 \left\| \sqrt{B^2+C}u \right\|, \tag{3.13}$$

for all $u \in \mathcal{D}(\sqrt{B^2+C})$. Now, both B and $-B$ satisfy (3.13), in virtue of [13], Theorem 2.4, p. 499, we obtain the expected results.

(2) From the previous statement and Theorem 3.35 of [13], p. 281.

(3) Thanks to Lemma 3.7, $\sqrt{B^2+C}$ is an invertible m-$(\pi/4)$-accretive operator. Thus, due to [25], example 2, p. 431, it follows that

$$\sqrt{B^2+C} \in \text{ BIP}\,(\mathcal{H}, \pi/4).$$

Therefore, from [26], Proposition 3.1, p. 170, we obtain the expected result.

□

Remark 3.9. Assume that (H_1) and (H_2) hold. Then

(1) We have $\mathcal{D}(L) = \mathcal{D}(B) \cap \mathcal{D}(\sqrt{B^2+C}) = \mathcal{D}(M)$, thus

$$\mathcal{D}(L) = \mathcal{D}(L) \cap \mathcal{D}(M) = \mathcal{D}(L+M) \subset \mathcal{D}(B),$$

and

$$\mathcal{D}(L) = \mathcal{D}(L) \cap \mathcal{D}(M) = \mathcal{D}(L-M) \subset \mathcal{D}(\sqrt{B^2+C}),$$

hence, due to (H_3), we obtain that

$$\mathcal{D}(L) = \mathcal{D}(M) = \mathcal{D}(B) \cap \mathcal{D}(\sqrt{B^2+C}) = \mathcal{D}(\sqrt{B^2+C}).$$

(2) From (H_4), L and M are resolvent commuting. Moreover, since $\mathcal{D}(L) = \mathcal{D}(M)$, we deduce that $LM = ML$.

(3) From (H_2), since $L - M = 2\sqrt{B^2+C}$, it follows that $0 \in \rho(L-M)$. But, since B is not supposed to invertible, in general $0 \notin \rho(L+M)$.

(4) Assumption (H_5) holds if one of the following conditions is satisfied

(i) $\left\| B\sqrt{B^2+C}^{-1} \right\| < 1.$

(ii) There exists $c > 1$, such that

$$\mathrm{Re}\left(\left\langle \sqrt{B^2+C}u, Bu \right\rangle\right) \geq c\left\|Bu\right\|^2, \quad \text{for all } u \in \mathcal{D}(\sqrt{B^2+C}).$$

(iii) B is accretive.

(5) Thanks to (H_6), due to [2], Theorem 5, p. 526, since $-M = \sqrt{B^2+C} - B$ is m-accretive, then $-M$ is the infinitesimal generator of a cosine function $(Cos(t))_{t\in\mathbb{R}}$, with $Cos(t) = \cos(t\sqrt{-M})$, and the associated sine function $(Sin(t))_{t\in\mathbb{R}}$, with $Sin(t) = \sin(t\sqrt{-M})$. Furthermore, due to [2], p. 518, we have

$$\left\|\cos(t\sqrt{-M})\right\| \leq me^{r|t|}, \quad \text{for all } t \in \mathbb{R} \text{ and some } r \geqslant 0, m > 0.$$

(6) We can derive a similar result to Theorem 3.4 for assumptions (**A.1**)-(**A.7**), if we assume that $B^2 - \gamma_1 I$ is m-accretive and $C - \gamma_2 I$ is accretive for some $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 + \gamma_2 > 0$. Also, we can interchanging the role of B^2 and C in all assumptions (**A.1**)-(**A.4**), (**A.6**)-(**A.7**), taking in account that B^2 is accretive (resp. $B^2 - \gamma I$ is accretive with some $\gamma > 0$) and $C - \gamma I$ is m-accretive with some $\gamma > 0$ (resp. C is m-accretive) such that $\mathcal{D}(C) \subset \mathcal{D}(B^2)$, by being careful with domains. In this case we have $\mathcal{D}(C) \subset \mathcal{D}(B^2) \subset \mathcal{D}(B)$. For what assumption (**A.5**) becomes:

$$B \text{ is an accretive operator with } \mathcal{D}(C) = \mathcal{D}(B).$$

(7) In Theorem 3.4, if assumption (**A.5**) holds, then it involves (H_5).

(8) In Theorem 3.4, if we assume that B and C are bounded operators, then $B^2 + C$ is m-accretive as an accretive bounded operator and also invertible. Hence, (H_3) is automatically satisfied. Moreover, to verify (H_4), it suffices to assume that B and C commute. Finally, from Lemma 3.8, since $W(-M)$ is bounded, (H_6) holds.

3.3. **Proof to Theorem 3.1.**

Assume that (3.5) holds. Then, we have to find a classical solution u to problem (3.4). To this end, we set

$$v(\cdot) = u''(\cdot) - Mu(\cdot).$$

Moreover, for $x \in [a, b]$, we have

$$\begin{array}{rcl} v''(x) & = & u^{(4)}(x) - Mu''(x) \\ & = & (L+M)u''(x) - Mu''(x) - LMu(x) + f(x) \\ & = & L(u''(x) - Mu(x)) + f(x) \\ & = & Lv(x) + f(x). \end{array}$$

Then, there exists a unique solution to problem (3.4) if and only if there exists a unique solution to the following hyperbolic problem

$$\left\{ \begin{array}{l} w''(x) - Mw(x) = v(x), \quad x \in (a,b) \\ w(a) \ = \ u_1, \ \ w'(a) \ = \ u_2, \end{array} \right. \tag{3.14}$$

where the second member v is the unique solution to the elliptic problem

$$\left\{ \begin{array}{l} v''(x) - Lv(x) = f(x), \quad x \in (a,b) \\ v(a) \ = \ u_3 - Mu_1, \ \ v(b) \ = \ u_4. \end{array} \right. \tag{3.15}$$

From Lemma 3.8 and Remark 3.9, using [7], p. 169-170, there exists a unique solution v to problem (3.15), given by (3.7), such that

$$v(\cdot) \in W^{2,p}(a,b;\mathcal{H}) \cap L^p(a,b;\mathcal{D}(L)). \tag{3.16}$$

Moreover, since $W^{2,p}(a,b;\mathcal{H}) \hookrightarrow C^1(a,b;\mathcal{H})$, with continuous injection, it follows that $v(\cdot) \in C^1(a,b;\mathcal{H})$ and due to Remark 3.7, from [30], Proposition 2.4, p. 79, there exists a unique solution w to problem (3.14) if and only if

$$w(a) \in \mathcal{D}(M) \quad \text{and} \quad w'(a) \in \{\varphi \in \mathcal{H} : Cos(t)\varphi \in C^1(a,b;\mathcal{H})\}.$$

Thus, due to (3.5), from [30], (2.19) in Proposition 2.2, p. 77, it follows that there exists a unique solution w to problem (3.14) given by (3.6). Note that by uniqueness, $w = F_U$.

Then, from (3.16), we deduce that

$$-(L+M)u''(\cdot) + LMu(\cdot) \in W^{2,p}(a,b;\mathcal{H}) \subset L^p(a,b;\mathcal{H}),$$

hence, since

$$u^{(4)}(\cdot) = (L+M)u''(\cdot) - LMu(\cdot) + f(\cdot) \in L^p(a,b;\mathcal{H}),$$

we obtain that (3.3) holds.

Conversely, assume that there exists a unique solution u to problem (3.4) such that (3.3) holds. Then from [9], Teorema 2', p. 678, we obtain that (3.5) holds. For the reader convenience, note that this result has been recall more recently in English in [29], Lemma 3.1, p. 638.

3.4. **Proof to Theorem 3.3.**

Let u be a classical solution to (3.2). From Theorem 3.3, there exists a unique classical solution F_0 to problem (3.4). Note that F_0 corresponds to F_U with $U = (0,0,0,0)$.

Set $u_h := u - F_0$. Thus, u_h is a classical solution to

$$u_h^{(4)}(x) - (L+M)u_h''(x) + LMu_h(x) = 0, \quad a.e.\ x \in (a,b).$$

Now, it remains to determine explicitly u_h. To this end, we will solve the previous homogeneous equation. Due to Remark 3.7, since $0 \in \rho(L-M)$, we set

$$\left\{\begin{array}{lcll} v(\cdot) & := & L(L-M)^{-1}u_h(\cdot) - (L-M)^{-1}u_h''(\cdot) & \in L^p\left(a,b;\mathcal{D}(M)\right) \\ w(\cdot) & := & -M(L-M)^{-1}u_h(\cdot) + (L-M)^{-1}u_h''(\cdot) & \in L^p\left(a,b;\mathcal{D}(L)\right). \end{array}\right. \tag{3.17}$$

Since $u_h(\cdot) \in W^{4,p}(a,b;\mathcal{H}) \cap L^p(a,b;\mathcal{D}(LM))$ with $u_h''(\cdot) \in L^p(a,b;\mathcal{D}(L+M))$, it follows that

$$v(\cdot) \in W^{2,p}(a,b;\mathcal{H}) \cap L^p(a,b;\mathcal{D}(M)) \quad \text{and} \quad w(\cdot) \in W^{2,p}(a,b;\mathcal{H}) \cap L^p(a,b;\mathcal{D}(L)),$$

with

$$v''(x) - Mv(x) = -(L-M)^{-1}\left(u_h^{(4)}(x) - (L+M)u_h''(x) + LMu_h(x)\right) = 0,$$

and

$$w''(x) - Lw(x) = (L-M)^{-1}\left(u_h^{(4)}(x) - (L+M)u_h''(x) + LMu_h(x)\right) = 0.$$

From Lemma 3.8 and Remark 3.9, due to the proof of Theorem 5, p. 173, in [7] or (18) in [8], the solution $w(\cdot)$ of the previous homogeneous equation reads as

$$w(x) = e^{(x-a)\sqrt{L}}K_3 + e^{(b-x)\sqrt{L}}K_4, \quad \text{for all } x \in [a,b],$$

where $K_3, K_4 \in \mathcal{H}$.

Moreover, due to Remark 3.7, from [1], Corollary 3.14.8, p. 209, the solution $v(\cdot)$ of the previous homogeneous equation reads as

$$v(x) = Cos(x-a)K_1 + Sin(x-a)K_2,$$

where $K_1, K_2 \in \mathcal{H}$.

We also refer to [28], Theorem 1.1, p. 591 and remark at the top of p. 592, to justify the expression of $v(\cdot)$.

Finally, since

$$v(\cdot) + w(\cdot) = (L-M)(L-M)^{-1}u_h(\cdot) = u_h(\cdot),$$

we conclude that, for all $x \in [a,b]$, we have

$$u_h(x) = Cos(x-a)K_1 + Sin(x-a)K_2 + e^{(x-a)\sqrt{L}}K_3 + e^{(b-x)\sqrt{L}}K_4.$$

Thus, $u(\cdot)$ satisfies (3.8), which gives the result.

3.5. **Proof to Theorem 3.4.** It suffices to prove that Lemma 3.7 remains valid under each one of this assumptions.

(1) Assume (**A.1**). The proof is inspired from [22] and [20]. Let $\gamma > 0$. Note first that $B^2 + C - \gamma I$ with domain $\mathcal{D}(B^2)$ is accretive and densely defined. So, by Proposition 2.4, to prove that $B^2 + C - \gamma I$ is m-accretive, it suffices to show that $B^2 + C - \gamma I$ is closed and its adjoint is accretive.

Let $u \in \mathcal{D}(B^2)$, we have

$$\begin{array}{rcl} \mathrm{Re}\left(\langle B^2u, (C-\gamma I)u\rangle\right) & = & \mathrm{Re}\left(\langle B^2u, Cu\rangle\right) - \gamma \mathrm{Re}\left(\langle B^2u, u\rangle\right) \\ & \geq & \mathrm{Re}\left(\langle B^2u, Cu\rangle\right) - \gamma \left\|B^2u\right\| \left\|u\right\|, \end{array}$$

it follows from (3.9) that

$$\mathrm{Re}\left(\left\langle (B^2+C-\gamma I)u, B^2u\right\rangle\right) \geq \mathrm{Re}\left(\left\langle B^2u, Cu\right\rangle\right) - \gamma \left\|B^2u\right\| \left\|u\right\| + \left\|B^2u\right\|^2,$$

hence

$$\begin{array}{rcl} \mathrm{Re}\left(\langle (B^2+C-\gamma I)u, B^2u\rangle\right) & \geq & (1-\beta)\left\|B^2u\right\|^2 - \alpha\left\|u\right\|^2 \\ & & -(\gamma+\delta)\left\|B^2u\right\| \left\|u\right\|. \end{array} \tag{3.18}$$

Thus

$$(1-\beta)\left\|B^2u\right\|^2 \leq \left[(\gamma+\delta)\left\|u\right\| + \left\|(B^2+C-\gamma I)u\right\|\right]\left\|B^2u\right\| + \alpha\left\|u\right\|^2.$$

Solving this inequality by taking $\|B^2u\|$ as a variable, we obtain

$$\left\|B^2u\right\| \leq \frac{1}{1-\beta}\left\|(B^2+C-\gamma I)u\right\| + \kappa\left\|u\right\|, \tag{3.19}$$

for all $u \in \mathcal{D}(B^2)$, with $\kappa = \dfrac{\gamma+\delta+\sqrt{\alpha(1-\beta)}}{1-\beta}$. On the other hand, since $\mathcal{D}(B^2) \subset \mathcal{D}(C) = \mathcal{D}(C-\gamma I)$, with $\mathcal{D}(B^2)$ dense in $\mathcal{H}$, there exists a constant $\vartheta > 0$, such that

$$\|(C-\gamma I)u\| \leq \vartheta(\|u\| + \left\|B^2u\right\|), \tag{3.20}$$

for all $u \in \mathcal{D}(B^2)$. Now, let a sequence $(u_n)_n \subset \mathcal{D}(B^2)$ such that $u_n \longrightarrow u$ and $(B^2 + C - \gamma I)u_n \longrightarrow v$. Let $n \neq m$, by (3.19), we have

$$\left\|B^2(u_n - u_m)\right\| \leq \frac{1}{1-\beta}\left\|(B^2 + C)(u_n - u_m)\right\| + \kappa \left\|u_n - u_m\right\|.$$

Therefore, the sequence $(B^2 u_n)_n$ converge. Since B^2 is closed we conclude that $B^2 u_n \longrightarrow B^2 u$ and $u \in \mathcal{D}(B^2)$. By (3.20), we have

$$\|(C - \gamma I)(u_n - u)\| \leq \vartheta \|u_n - u\| + (1 + \vartheta)\left\|B^2(u_n - u)\right\|.$$

Hence $(B^2+C-\gamma I)u_n \longrightarrow (B^2+C-\gamma I)u$ and $v = (B^2+C-\gamma I)u$ with $u \in \mathcal{D}(B^2)$. This prove that $B^2 + C - \gamma I$ is closed, densely defined and accretive.

Now we show that the adjoint $(B^2 + C - \gamma I)^*$ of $(B^2 + C - \gamma I)$ is accretive. It follows from (3.18) that

$$\mathrm{Re}\left(\left\langle (B^2 + C - \gamma I)u, B^2 u\right\rangle\right) + \alpha \|u\|^2 + (\gamma + \delta)\left\|B^2 u\right\| \|u\| \geq 0,$$

for all $u \in \mathcal{D}(B^2)$. Hence we obtain

$$\begin{aligned}\mathrm{Re}&\left(\left\langle (B^2 + C - \gamma I)u, \left(I + \frac{1}{n}B^2\right)u\right\rangle\right)\\ &+\frac{1}{n}\alpha \|u\|^2 + \frac{1}{n}(\gamma + \delta)\|B^2 x\| \|u\| \geq 0,\end{aligned}$$

for all $n \in \mathbb{N} \setminus \{0\}$. Now let $v \in \mathcal{H}$. Then for $u = \left(I + \frac{1}{n}B^2\right)^{-1} v \in \mathcal{D}(B^2)$, we have

$$\begin{aligned}&\mathrm{Re}\left(\left\langle (B^2 + C - \gamma I)\left(I + \frac{1}{n}B^2\right)^{-1} v, v\right\rangle\right)\\ &+\frac{1}{n}\alpha \|v\|^2 + \frac{1}{n}(\gamma + \delta)\left\|B^2\left(I + \frac{1}{n}B^2\right)^{-1} v\right\| \|v\| \geq 0,\end{aligned}$$

for all $v \in \mathcal{H}$, $n \in \mathbb{N} \setminus \{0\}$. In particular, for $v \in \mathcal{D}((B^2 + C - \gamma I)^*)$, we obtain

$$\begin{aligned}&\mathrm{Re}\left(\left\langle \left(I + \frac{1}{n}B^2\right)^{-1} v, (B^2 + C - \gamma I)^* v\right\rangle\right)\\ &+\frac{1}{n}\alpha \|v\|^2 + \frac{1}{n}(\gamma + \delta)\left\|B^2\left(I + \frac{1}{n}B^2\right)^{-1} v\right\| \|v\| \geq 0,\end{aligned}$$

for all $n \in \mathbb{N}\setminus\{0\}$. Letting n to $+\infty$, we conclude that $(B^2+C-\gamma I)^*$ is accretive.

(2) Assume (**A.2**). We have, for $u \in \mathcal{D}(B^2)$,

$$\|(C - \gamma I)u\|^2 = \|Cu\|^2 + \gamma^2 \|u\|^2 - 2\mathrm{Re}\left(\langle Cu, u\rangle\right).$$

Since $C - \gamma I$ is accretive; then C is also accretive, hence

$$\|(C - \gamma I)u\|^2 \leq \|Cu\|^2 + \gamma^2 \|u\|^2,$$

for all $u \in \mathcal{D}(B^2)$. Now, using (3.10), we obtain

$$\|(C - \gamma I)u\|^2 \leq b\left\|B^2 u\right\|^2 + (\gamma^2 + a)\|u\|^2,$$

for all $u \in \mathcal{D}(B^2)$. This implies that $C-\gamma I$ is B^2-bounded with lower bound lesser than 1. By Theorem 2 in [10] we deduce that $B^2+C-\gamma I$ is m-accretive. Also (3.10) implies (3.9) in the case of $\delta = 0$, see [20], Remark 4.4. Thus the assumption (3.9) is stronger than the relative boundedness.

(3) Assume (**A.3**). Then, $I+(C-\gamma I)(B^2+t_0I)^{-1}$ is boundedly invertible, for some $t_0>0$.

Indeed, since $B^2+C-\gamma I$ is densely defined and accretive, it suffices to show that $\mathcal{R}(B^2+C-\gamma I+t_0I)=\mathcal{H}$. But, this follows immediately from

$$B^2+C-\gamma I+t_0I=(I+(C-\gamma I)(B^2+t_0I)^{-1})(B^2+t_0I),$$

and clearly $B^2+C-\gamma I+t_0I$ is invertible.

(4) Assume (**A.4**). Due to Proposition 2.12, p. 55 in [32], the lower bound β in (3.10) is equal to $\sup\limits_{t>0}\left\|C(B^2+tI)^{-1}\right\|$. Hence (**A.4**) implies (**A.2**), see [21], Theorem 1, p. 851.

(5) Assume (**A.5**). Since B is an accretive operator, from Theorem 1.2 in [12], we have for an arbitrary $\nu>0$,

$$\|Bu\|^2 \leq \nu\|u\|^2+\frac{1}{\nu}\left\|B^2u\right\|^2, \tag{3.21}$$

for all $u \in \mathcal{D}(B^2)$. On the other hand, since $\mathcal{D}(B^2)\subset\mathcal{D}(B)\subset\mathcal{D}(C-\gamma I)$, with $\mathcal{D}(B^2)$ dense in $\mathcal{H}$, there exists a constant $\vartheta_1>0$, such that

$$\|(C-\gamma I)u\|^2 \leq \vartheta_1(\|u\|^2+\|Bu\|^2), \tag{3.22}$$

for all $u\in\mathcal{D}(B)$. Now, by (3.21) and (3.22) we obtain that

$$\|(C-\gamma I)u\|^2 \leq \vartheta_1(1+\nu)\|u\|^2+\frac{\vartheta_1}{\nu}\left\|B^2u\right\|^2, \tag{3.23}$$

for all $u\in\mathcal{D}(B^2)$ and all arbitrary $\nu>0$. Choosing $\nu>0$ large enough, we obtain that $\dfrac{\vartheta_1}{\nu}<1$. Thus, $C-\gamma I$ is B^2-bounded with lower bound lesser than 1. Then, $B^2+C-\gamma I$ with domain $\mathcal{D}(B^2)$ is m-accretive.

(6) Assume (**A.6**). From Theorem 6.10 in [23], there exists a constant $\kappa>0$ such that for every $u\in\mathcal{D}(B^2)$ and an arbitrary $\rho>0$, we have

$$\left\|(B^2)^\alpha u\right\|^2 \leq \kappa(\rho^\alpha\|u\|^2+\frac{1}{\rho^{1-\alpha}}\left\|B^2u\right\|^2).$$

Since $\mathcal{D}((B^2)^\alpha)\subset\mathcal{D}(C)=\mathcal{D}(C-\gamma I)$, with $\mathcal{D}((B^2)^\alpha)$ dense in $\mathcal{H}$, there exists a constant $\upsilon>0$, such that

$$\|(C-\gamma I)u\|^2 \leq \upsilon\left\|(B^2)^\alpha u\right\|^2,$$

for all $u\in\mathcal{D}((B^2)^\alpha)$. It follows that

$$\|(C-\gamma I)u\|^2 \leq \upsilon\kappa(\rho^\alpha\|u\|^2+\frac{1}{\rho^{1-\alpha}}\left\|B^2u\right\|^2),$$

for all $x\in\mathcal{D}(B^2)$. Choosing $\rho>0$ large enough, we obtain that $\dfrac{\upsilon\kappa}{\rho^{1-\alpha}}<1$. Thus, $C-\gamma I$ is B^2-bounded with bound less than 1.

(7) Assume (**A.7**). In this case C is B^2-bounded with lower bound equal zero.

3.6. **Proof of Theorem 3.5.**

By definition $C-\gamma I$ is m-ω_1-accretive operator, means that $e^{\pm i\theta}C-\gamma I$ are also m-accretive for $\theta=\dfrac{\pi}{2}-\omega_1$ with $0\leq\omega_1\leq\dfrac{\pi}{2}$.

In fact that B is an ϑ-accretive operator in $\mathcal{H}$ and fulfill the requirements of Theorem 4.1, p. 699 in [11], then B^2 is ω_2-accretive with $\omega_2\leq 2\arctan(c)$, $c\leq 1$.

Then $e^{\pm i\theta}B^2$ are also accretive for $\theta=\dfrac{\pi}{2}-\omega_2$, $0\leq\omega_2\leq 2\arctan(c)$. Since all the inequalities given in (**A.1**)-(**A.6**) are invariant up to the factor $e^{i\theta}$ for all θ, it follows that $e^{\pm i\theta}C+e^{\pm i\theta}B^2-\gamma I$ are m-accretive for all $\theta=\dfrac{\pi}{2}-\omega$ with $0\leq\omega\leq\max\{\omega_1,\omega_2\}$. This implies that $B^2+C-\gamma I$ is also m-ω-accretive with $\omega=\max\{\omega_1,\omega_2\}$.

4. An example

We consider the following fourth order linear partial differential problem:

$$(E)\begin{cases}\dfrac{\partial^4 u}{\partial x^4}(x,y)-2ip_0(y)\dfrac{\partial^3 u}{\partial y\partial x^2}(x,y)-2ip_1(y)\dfrac{\partial^2 u}{\partial x^2}(x,y)-\alpha p_0(y)\dfrac{\partial u}{\partial y}(x,y)\\ \qquad -(\alpha p_1(y)+\beta)u(x,y)+\gamma u(x,y)=f(x,y),\quad x,y\in(0,1)\\ u(0,y)=u(1,y)=\dfrac{\partial u}{\partial x}(0,y)=0,\quad y\in(0,1)\\ \dfrac{\partial^2 u}{\partial x^2}(0,y)=\dfrac{\partial^2 u}{\partial x^2}(1,y)=0,\quad y\in(0,1)\\ u(x,0)=u(x,1)=0,\quad x\in(0,1)\\ \dfrac{\partial^2 u}{\partial x^2}(x,0)=\dfrac{\partial^2 u}{\partial x^2}(x,1)=0,\quad x\in(0,1)\end{cases}$$

where,

- $\alpha\in\mathbb{R}$, $\beta\in\mathbb{C}$, $p_0,p_1\in C^1(0,1)$ and $p_0(x)\neq 0$ for all $x\in[0,1]$,
- $\gamma=-\left(\dfrac{r+1}{4\varepsilon}m_1+m_2\right)$, with $r,\varepsilon>0$ chosen such that $m_0-\varepsilon(1+r)m_1>0$, for some non-negative constants m_0, m_1 and m_2 described below,
- $f\in L^2((0,1)^2;\mathbb{C})$.

We set

$$\begin{cases}\mathcal{D}(B)=\{\psi\in H^1(0,1):\psi(0)=\psi(1)=0\}\\ B\psi=p_0\dfrac{\partial\psi}{\partial y}+p_1\psi,\quad\text{for all}\,\psi\in\mathcal{D}(B),\end{cases}$$

and

$$\begin{cases}\mathcal{D}(C)=\{\phi\in H^1(0,1):\phi(0)=\phi(1)=0\}\\ C\phi=\alpha p_0\dfrac{\partial\phi}{\partial y}+(\alpha p_1+\beta)\phi,\quad\text{for all}\,\phi\in\mathcal{D}(C).\end{cases}$$

Note that such operators are well-defined, see for instance [13], Example 2.6, p. 144.

Then, the fourth order differential problem (E) reads as

$$\begin{cases} u^{(4)}(x) - 2iBu''(x) - (C-\gamma I)u(x) = f(x), & x \in (0,1) \\ u(0) \ = u'(0) \ = 0 \\ u''(0) = u''(1) = 0, \end{cases} \tag{4.1}$$

where $u(x) := u(x)(y)$ and $f(x) := f(x)(y)$, with $f \in L^2(0,1;\mathcal{H})$ and $\mathcal{H} = L^2(0,1;\mathbb{C})$. In view to use Theorem 3.1, the boundary condition $u''(1) - Mu(1) = 0$ reads as $u''(1) = 0$.

The following result states the existence and uniqueness of the solution to problem (4.1).

Theorem 4.1. *Let $\mathcal{H} = L^2(0,1;\mathbb{C})$. Assume that*

(1) $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$, $p_0 \in C^2(0,1)$, $p_1 \in C^1(0,1)$ *and* $p_0(x) \neq 0$ *for all* $x \in [0,1]$,

(2) $p_1 - \frac{1}{2}p_0' \geq 0$ *and* $\alpha p_1 + \mathrm{Re}(\beta) - \frac{\alpha}{2}p_0' \geq 0$,

(3) *there exist some non-negative constants m_0, m_1 and m_2 given by*

$$p_0^2 > m_0 > 0, \qquad |p_0|\,|p_0' - 2p_1| \leq m_1 \quad \text{and} \quad \left|p_1^2 + p_0p_1'\right| \leq m_2, \tag{4.2}$$

such that $\gamma = -\left(\frac{r+1}{4\varepsilon}m_1 + m_2\right)$, *with* $r,\varepsilon > 0$ *such that* $m_0 - \varepsilon(1+r)m_1 > 0$.

(4) $B\sqrt{-(B^2 - C + \gamma I)} = \sqrt{-(B^2 - C + \gamma I)}B$.

(5) $0 \in \rho(iB + \sqrt{-(B^2 - C + \gamma I)})$.

(6) *There exists $\nu > 0$ such that*

$$W(\sqrt{-(B^2 - C + \gamma I)} - iB) \subset \{z \in \mathbb{C} : \quad 4\nu^2\mathrm{Re}(z) \geqslant \mathrm{Im}(z)^2\}.$$

Then, there exists a unique classical solution $u(\cdot)$ to problem (4.1). *Moreover, $u(\cdot)$ is determined as in Theorem 3.1.*

Proof. In order to use Corollary 3.2, we need first to verify assumptions of Theorem 3.3. The proof is split into several steps:

(1) *We prove that $-B^2 - \gamma I$ is m-ω-accretive, with* $\omega = \arctan\left(\frac{1}{r}\right)$.

For $\psi \in \mathcal{D}(B^2) \subset \{\psi \in H^2(0,1) : \psi(0) = \psi(1) = 0\} \subset \mathcal{D}(B)$, we have

$$-B^2\psi = -p_0^2\psi'' - p_0(p_0' + 2p_1)\psi' - (p_1^2 + p_0p_1')\psi.$$

Thus, there exists a nonegative constants m_0, m_1 and m_2 such that (4.2) holds. From Example V-3.34, p. 280 in [13], operator B^2 is m-ω-accretive operator with vertex γ, where $\gamma = -\left(\frac{r+1}{4\varepsilon}m_1 + m_2\right)$, $\omega = \arctan\left(\frac{1}{r}\right)$, $r,\varepsilon > 0$ chosen such that

$$\begin{cases} 0 < \varepsilon < \dfrac{m_0}{m_1(1+r)}, & \text{if } m_1 > 0, \\ 0 < \varepsilon, & \text{if } m_1 = 0. \end{cases}$$

Hence operator $-B^2 - \gamma I$ is m-ω-accretive, with $\omega = \arctan\left(\frac{1}{r}\right)$.

(2) *Now, we show that C is accretive.*

Let $\phi \in \mathcal{D}(C)$, we have

$$\langle C\phi, \phi\rangle = \alpha \int_0^1 p_0(y)\phi'(y)\overline{\phi(y)}dy + \int_0^1 (\alpha p_1(y) + \beta)\,|\phi(y)|^2\,dy.$$

By integration by parts,

$$\langle C\phi, \phi\rangle = -\alpha \int_0^1 p_0(y)\phi(y)\overline{\phi'(y)}dy + \int_0^1 (\alpha p_1(y) + \beta - \alpha p_0'(y))\,|\phi(y)|^2\,dy.$$

Also

$$\overline{\langle C\phi, \phi\rangle} = \langle \phi, C\phi\rangle = \alpha \int_0^1 p_0(y)\phi(y)\overline{\phi'(y)}dy + \int_0^1 (\alpha p_1(y) + \overline{\beta})\,|\phi(y)|^2\,dy.$$

Thus

$$\mathrm{Re}\,\langle C\phi, \phi\rangle = \int_0^1 (\alpha p_1 + Re(\beta) - \frac{\alpha}{2}p_0')\,|\phi(y)|^2\,dy.$$

Since $\alpha p_1 + \mathrm{Re}(\beta) - \dfrac{\alpha}{2}p_0' \geq 0$, then C is accretive.

(3) *In particular, B is accretive.*

Indeed, if we take $\alpha = 1$ and $\beta = 0$ in the previous proof, we obtain

$$\mathrm{Re}\,\langle B\phi, \phi\rangle = \int_0^1 (p_1 - \frac{1}{2}p_0')\,|\phi(y)|^2\,dy.$$

Now, since $p_1 - \dfrac{1}{2}p_0' \geq 0$, then B is accretive.

(4) *Then, we show that $-(B^2 - C + \gamma I)$ with domain $\mathcal{D}(B^2)$ is m-accretive. Also, we prove that $-(B^2 - C + \gamma I)$ admits an unique square root $\sqrt{-(B^2 - C + \gamma I)}$ which is m-$\pi/4$-accretive with*

$$\mathcal{D}(\sqrt{-(B^2 - C + \gamma I)}) = \{\psi \in H^1(0,1) : \psi(0) = \psi(1) = 0\}.$$

From statement (1), $-(B^2 + \gamma I)$ with domain $\mathcal{D}(B^2)$ is m-accretive. Due to statement (2), C is accretive and from [18],

$$\mathcal{D}(\sqrt{-(B^2 + \gamma I)}) = \{\psi \in H^1(0,1) : \psi(0) = \psi(1) = 0\} = \mathcal{D}(C).$$

Now the desired result holds from Theorem 3.4 with (**A.6**). Therefore, due to [18], we have

$$\mathcal{D}(\sqrt{-(B^2 - C + \gamma I)}) = \{\psi \in H^1(0,1) : \psi(0) = \psi(1) = 0\}.$$

(5) *It remains to prove that $-(B^2 - C + \gamma I)^{-1}$ exists and is bounded.*

As previously, for $\psi \in \mathcal{D}(B^2) \subset \{\psi \in H^2(0,1) : \psi(0) = \psi(1) = 0\} \subset \mathcal{D}(B)$, we have

$$[B^2 - C + \gamma I]\psi = p_0^2\psi'' + (p_0p_0' + p_1(2p_0 - \alpha))\psi' + (p_1^2 + p_1'(p_0 - \alpha) - \beta + \gamma)\psi.$$

Since $p_0 \in C^2(0,1)$ and $p_1 \in C^1(0,1)$, it follows that p_0'' and p_1' are continuous on $[0,1]$. Thus, $2(p_0''p_0 + (p_0')^2)$, $p_0'(p_0' + 2p_1) + p_0p_0'' + p_1'(2p_0 - \alpha)$ and $p_1^2 + p_1'(p_0 - \alpha) - \beta + \gamma$ are continuous on $[0,1]$. By a similar way as in [13], Section 3-III, p. 146-149, we prove that $-(B^2 - C + \gamma I)^{-1}$ exists and is bounded.

(6) *Finally,* $L = iB + \sqrt{-(B^2 - C + \gamma I)}$ *and* $-M = -(iB - \sqrt{-(B^2 - C + \gamma I)})$ *with domain* $\mathcal{D}(\sqrt{-(B^2 - C + \gamma I)}) \subset \mathcal{D}(B)$ *are m-accretive operators.*
See Lemma 3.8.

□

Remark 4.2. In Theorem 4.1, assumption (6) holds when p_0, α, p_1 are positive constants and $\beta = 0$. Further, we can see that if $\alpha = 1$, then $B = C$ so assumption (4) holds. Now, by choosing appropriate constants p_0, p_1 such that

$$\left\| B(\sqrt{-(B^2 - C + \gamma I)})^{-1} \right\| < 1,$$

then assumption (5) holds.

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