Solvability of a fourth order elliptic problem in a bounded sector, part II

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Abstract

After different variables and functions changes, the generalized dispersal problem, recalled in (1) below and considered in part I, see Labbas, Maingot and Thorel [14], leads us to consider, to study and to invert the sum of linear operators (4) below in a suitable Banach space by using two strategies: namely the theory of sums of operators in Banach spaces as developed by Da Prato-Grisvard [4] and successfully improved by Dore-Venni [5].
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1 Introduction and main result

This work is a natural continuation of Labbas, Maingot and Thorel [14], where we have considered, for $k > 0$, the following problem

$$\left\{\begin{array}{ll} \Delta^2 u - k\Delta u = f & \text{in } S_{\omega,\rho} \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma_0 \cup \Gamma_{\omega,\rho} \\ u = \dfrac{\partial^2 u}{\partial n^2} = 0 & \text{on } \Gamma_\rho, \end{array}\right. \tag{1}$$

with

$$\left\{\begin{array}{lcl} S_{\omega,\rho} & = & \{(x,y) = (r\cos\theta, r\sin\theta) : 0 < r < \rho \text{ and } 0 < \theta < \omega\} \\ \Gamma_0 & = & (0,\rho)\times\{0\} \\ \Gamma_\omega & = & \{(r\cos\omega, r\sin\omega) \ : \ 0 < r < \rho\} \\ \Gamma_\rho & = & \{(\rho\cos\theta, \rho\sin\theta) \ : \ 0 < \theta < \omega\}, \end{array}\right.$$

for given $\rho > 0$ and $\omega \in (0, 2\pi]$.

In part I, see Labbas, Maingot and Thorel [14], to study problem (1), we have done many variables and functions changes to write it as a sum of linear operators. To this end, we have introduced the following functions for $t > 0$ and $(r,\theta) \in S_{\rho,\omega}$:

$$\left\{\begin{array}{l} v(r,\theta) = u(r\cos\theta, r\sin\theta) \\ G(t)(\theta) := G(t,\theta) = g(\rho e^{-t},\theta) = f\left(\rho e^{-t}\cos\theta, \rho e^{-t}\sin\theta\right) \\ \phi(t)(\theta) := \phi(t,\theta) = \dfrac{v(\rho e^{-t},\theta)}{\rho e^{-t}} \\ H(t)(\theta) := H(t,\theta) = e^{-3t}G(t)(\theta), \end{array}\right. \tag{2}$$

and the two abstract vector-valued functions

$$V(t) = \begin{pmatrix} e^{\nu t}\phi(t) \\ e^{\nu t}\phi''(t) \end{pmatrix}, \quad \mathcal{F}_\nu(t) = \begin{pmatrix} 0 \\ \rho^3 e^{\nu t} H(t) \end{pmatrix}, \quad \nu = 3 - \frac{2}{p} \in (1,3) \text{ with } p > 1. \tag{3}$$

By considering the Banach space

$$X = W_0^{2,p}(0,\omega) \times L^p(0,\omega),$$

and after the changes indicated above in (2) and (3), we have rewritten problem (1) in the space $L^p(0,+\infty;X)$, see Labbas, Maingot and Thorel [14], in the following form

$$\left(\mathcal{L}_{1,\nu} + \mathcal{L}_2\right) V + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_{2,\nu}\right) V = \mathcal{F}_\nu,$$

where

$$\left\{\begin{array}{lll} D(\mathcal{L}_{1,\nu}) & = & \left\{V \in W^{2,p}(0,+\infty;X) : V(0) = V(+\infty) = 0\right\} \\ \left[\mathcal{L}_{1,\nu}(V)\right](t) & = & \left(\partial_t - \nu I\right)^2 V(t) = V''(t) - 2\nu V'(t) + \nu^2 V(t), \end{array}\right.$$

$$\left\{\begin{array}{lll} D(\mathcal{L}_2) & = & \{V \in L^p(0,+\infty;X) : \text{for } a.e.\ t \in (0,+\infty),\ V(t) \in D(\mathcal{A})\} \\ \left[\mathcal{L}_2(V)\right](t) & = & -\mathcal{A}V(t), \end{array}\right.$$

with

$$\left\{\begin{array}{lll} D(\mathcal{A}) & = & \left[W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right] \times W_0^{2,p}(0,\omega) \subset X \\ \mathcal{A}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & = & \begin{pmatrix} \psi_2 \\ -\left(\dfrac{\partial^2}{\partial\theta^2} + 1\right)^2 \psi_1 - 2\left(\dfrac{\partial^2}{\partial\theta^2} - 1\right)\psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}), \end{array}\right.$$

and

$$\left\{\begin{array}{lll} D(\mathcal{P}_1) & = & \{V \in L^p(0,+\infty;X) : \text{for } a.e.\ t \in (0,+\infty),\ V(t) \in D(\mathcal{A}_0)\} \\ \left[\mathcal{P}_1(V)\right](t) & = & -e^{-2t}\mathcal{A}_0 V(t), \end{array}\right.$$

with

$$\left\{\begin{array}{lll} D(\mathcal{A}_0) & = & W_0^{2,p}(0,\omega) \times L^p(0,\omega) = X \\ \mathcal{A}_0\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & = & \begin{pmatrix} 0 \\ \left(\dfrac{\partial^2}{\partial\theta^2} + 1\right)\psi_1 + \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}_0), \end{array}\right.$$

and

$$\left\{\begin{array}{lll} D(\mathcal{P}_{2,\nu}) & = & W^{1,p}(0,+\infty;X) \\ \left[\mathcal{P}_{2,\nu}(V)\right](t) & = & -e^{-2t}\left(\mathcal{B}_{2,\nu}V\right)(t), \end{array}\right.$$

with

$$\mathcal{B}_{2,\nu} = \begin{pmatrix} 0 & 0 \\ -2(\partial_t - \nu I) & 0 \end{pmatrix}.$$

In the present paper, we will focus ourselves on the resolution of the following abstract equation

$$\left(\mathcal{L}_{1,\mu} + \mathcal{L}_2\right) V + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_{2,\mu}\right) V = \mathcal{F}, \tag{4}$$

where $\mu \in \mathbb{R}$ is a general parameter and $\mathcal{F} \in L^p(0,+\infty;X)$.

Note that, in part I, see Labbas, Maingot and Thorel [14], subsection 3.4, we have worked with $\mu = \nu = 3 - \dfrac{2}{p}$ which comes from the variable change concerning the weighted Sobolev space. Here, in this second part, we consider a more general $\mu \in \mathbb{R}$.

The aim of this work is to show that there exists a unique classical solution of (4) that is a function V such that

$$V \in W^{2,p}(0,+\infty;X) \cap L^p(0,+\infty;D(\mathcal{A})).$$

This regularity is necessary to deduce all those of the function v stated in Theorem 2.2 in Labbas, Maingot and Thorel [14].

To this end, we will use the Da Prato-Grisvard sum theory in order to invert $\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2}$. Then, we solve the following equation

$$\left(\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2}\right)V + k\rho^2\left(\mathcal{P}_1+\mathcal{P}_{2,\mu}\right)V = \mathcal{F},$$

by using a perturbation argument. Next, we use some convexity inequalities to prove that V belongs to a more suitable space, more precisely

$$V \in W^{1,p}(0,+\infty;X)\cap L^p\left(0,+\infty;\left[W^{3,p}(0,\omega)\cap W_0^{2,p}(0,\omega)\right]\times L^p(0,\omega)\right).$$

At this step, V is the unique strong solution of (4), see (46). To obtain that V is a classical solution, we use the Dore-Venni sum theory, see Section 6.

Among the results that we will use is the fact that the roots of the equation

$$(\sinh(z)+z)(\sinh(z)-z)=0,$$

in $\mathbb{C}_+ := \{w\in\mathbb{C} : \mathrm{Re}(w)>0\}$, constitute a family of complex numbers $(z_j)_{j\geqslant 1}$ such that

$$\tau := \min_{j\geqslant 1}|\mathrm{Im}(z_j)|>0 \quad \text{and} \quad |z_j| \longrightarrow +\infty.$$

These roots are computed in Fädle [7] with $\tau \simeq 4.21239$.

Our main result is the following.

Theorem 1.1. Let $\mathcal{F}\in L^p(0,+\infty;X)$ and assume that

$$\omega\mu < \tau. \tag{5}$$

Then, there exists $\rho_0>0$ such that for all $\rho\in(0,\rho_0]$, the abstract equation

$$\left(\mathcal{L}_{1,\mu}+\mathcal{L}_2\right)V + k\rho^2\left(\mathcal{P}_1+\mathcal{P}_{2,\mu}\right)V = \mathcal{F},$$

has a unique classical solution $V\in L^p(0,+\infty;X)$, that is

$$V\in W^{2,p}(0,+\infty;X)\cap L^p(0,+\infty;D(\mathcal{A})).$$

In particular, $\mathcal{L}_{1,\mu}+\mathcal{L}_2$ is closed and $V\in D(\mathcal{L}_{1,\mu}+\mathcal{L}_2)$.

This second part is organized as follows. Section 2 is devoted to recalling some needed results. In Section 3, we analyze the spectral properties of operators $\mathcal{L}_{1,\mu}$ and $\mathcal{L}_2$ in view to study the invertibility of $\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2}$ in Section 4. In Section 5, by considering that operator $k\rho^2\left(\mathcal{P}_1+\mathcal{P}_{2,\mu}\right)$ is a perturbation, we deduce the existence and the uniqueness of a strong solution of equation (4). Finally, Section 6 is devoted to the proof of our main result given in Theorem 1.1.

2 Definitions and prerequisites

2.1 The class of Bounded Imaginary Powers of operators

Definition 2.1. A Banach space E is a UMD space if and only if for all $p\in(1,+\infty)$, the Hilbert transform is bounded from $L^p(\mathbb{R},E)$ into itself (see Bourgain [2] and Burkholder [3]).

Definition 2.2. Let $\alpha\in(0,\pi)$. Sect(α) denotes the space of closed linear operators T_1 which satisfying

$$\begin{array}{ll} i) & \sigma(T_1)\subset\overline{S_\alpha}, \\ ii) & \forall\,\alpha'\in(\alpha,\pi),\quad \sup\left\{\|\lambda(\lambda I-T_1)^{-1}\|_{\mathcal{L}(X)} : \ \lambda\in\mathbb{C}\setminus\overline{S_{\alpha'}}\right\}<+\infty, \end{array}$$

where

$$S_\alpha := \left\{\begin{array}{lll} \{z\in\mathbb{C} : z\neq 0 \ \text{ and } \ |\arg(z)|<\alpha\} & \text{if} & \alpha\in(0,\pi] \\ (0,+\infty) & \text{if} & \alpha=0, \end{array}\right. \tag{6}$$

see Haase [9], p. 19. Such an operator T_1 is called sectorial operator of angle α.

Remark 2.3. From Komatsu [11], p. 342, we know that any injective sectorial operator T_1 admits imaginary powers T_1^{is} for all $s \in \mathbb{R}$; but in general, T_1^{is} is not bounded.

Definition 2.4. Let $\kappa \in [0, \pi)$. We denote by $\mathrm{BIP}(E, \kappa)$, the class of sectorial injective operators T_2 such that

$$i) \quad \overline{D(T_2)} = \overline{R(T_2)} = E,$$

$$ii) \quad \forall\, s \in \mathbb{R}, \quad T_2^{is} \in \mathcal{L}(E),$$

$$iii) \quad \exists\, C \geqslant 1,\ \forall\, s \in \mathbb{R}, \quad ||T_2^{is}||_{\mathcal{L}(E)} \leqslant C e^{|s|\kappa},$$

see Prüss and Sohr [19], p. 430.

2.2 Recall on the sum of linear operators

Let us fix a pair of two closed linear densely defined operators $\mathcal{M}_1$ and $\mathcal{M}_2$ in a general Banach space $\mathcal{E}$. We note their domains by $D(\mathcal{M}_1)$ and $D(\mathcal{M}_2)$ respectively. Then we can define their sum by

$$\begin{cases} \mathcal{M}_1 w + \mathcal{M}_2 w \\ w \in D(\mathcal{M}_1) \cap D(\mathcal{M}_2). \end{cases}$$

We assume the following hypotheses

(H_1) There exist $\theta_{\mathcal{M}_1} \in [0, \pi)$, $\theta_{\mathcal{M}_2} \in [0, \pi)$, $C > 0$ and $R > 0$ such that

$$\begin{cases} \rho\left(\mathcal{M}_1\right) \supset \Sigma_{1,R} = \{z \in \mathbb{C} \setminus \{0\} : |z| \geqslant R \text{ and } |\arg(z)| < \pi - \theta_{\mathcal{M}_1}\} \\ \forall\, z \in \Sigma_{1,R}, \quad \left\|(\mathcal{M}_1 - zI)^{-1}\right\| \leqslant \dfrac{C}{|z|}, \end{cases}$$

and

$$\begin{cases} \rho\left(\mathcal{M}_2\right) \supset \Sigma_{2,R} = \{z \in \mathbb{C} \setminus \{0\} : |z| \geqslant R \text{ and } |\arg(z)| < \pi - \theta_{\mathcal{M}_2}\} \\ \forall\, z \in \Sigma_{2,R}, \quad \left\|(\mathcal{M}_2 - zI)^{-1}\right\| \leqslant \dfrac{C}{|z|}, \end{cases}$$

with

$$\theta_{\mathcal{M}_1} + \theta_{\mathcal{M}_2} < \pi.$$

(H_2) $\sigma(\mathcal{M}_1) \cap \sigma(-\mathcal{M}_2) = \emptyset$.

(H_3) The resolvents of $\mathcal{M}_1$ and $\mathcal{M}_2$ commute, that is

$$\left(\mathcal{M}_1 - \lambda_1 I\right)^{-1}\left(\mathcal{M}_2 - \lambda_2 I\right)^{-1} = \left(\mathcal{M}_2 - \lambda_2 I\right)^{-1}\left(\mathcal{M}_1 - \lambda_1 I\right)^{-1},$$

for all $\lambda_1 \in \rho\left(\mathcal{M}_1\right)$ and all $\lambda_2 \in \rho\left(\mathcal{M}_2\right)$.

Remark 2.5. Note that from (H_2), we have $\rho\left(\mathcal{M}_1\right) \cup \rho\left(-\mathcal{M}_2\right) = \mathbb{C}$ and in particular $\mathcal{M}_1$ or $\mathcal{M}_2$ is boundedly invertible.

Theorem 2.6 (Da Prato and Grisvard [4], Grisvard [8])**.** Assume that (H_1), (H_2) and (H_3) hold. Then, operator $\mathcal{M}_1 + \mathcal{M}_2$ is closable. Its closure $\overline{\mathcal{M}_1 + \mathcal{M}_2}$ is boundedly invertible and

$$\left(\overline{\mathcal{M}_1 + \mathcal{M}_2}\right)^{-1} = \frac{-1}{2i\pi} \int\limits_{\Gamma} \left(\mathcal{M}_1 - zI\right)^{-1}\left(\mathcal{M}_2 + zI\right)^{-1} dz; \tag{7}$$

where Γ is a path which separates $\sigma\left(\mathcal{M}_1\right)$ and $\sigma\left(-\mathcal{M}_2\right)$ and joins $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$ with θ_0 such that

$$\theta_{\mathcal{M}_1} < \theta_0 < \pi - \theta_{\mathcal{M}_2}.$$

This Theorem is proved in Da Prato and Grisvard [4] (Theorem 3.7, p. 324), when $R = 0$ and has been extended to the case $R \geqslant 0$ in Grisvard [8] (Theorem 2.1, p. 7). In this last case, the curve Γ does not need to be connected.

Corollary 2.7. Assume that (H_1), (H_2) and (H_3) hold. Let $i = 1, 2$ and $\mathcal{E}_i$ a Banach space with $D(\mathcal{M}_i) \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}$. We suppose that there exist $C > 0$ and $\delta \in (0,1)$ such that

$$\left\{ \begin{array}{l} \|w\|_{\mathcal{E}_i} \leqslant C \left(\|w\|_{\mathcal{E}} + \|w\|_{\mathcal{E}}^{1-\delta} \|\mathcal{M}_i w\|_{\mathcal{E}}^{\delta} \right) \\ \text{for every } w \in D(\mathcal{M}_i). \end{array} \right. \tag{8}$$

Then $D\left(\overline{\mathcal{M}_1 + \mathcal{M}_2}\right) \subset \mathcal{E}_i$.

Proof. It is enough to prove that the integral in (7) converges in $\mathcal{E}_1$. For all $\xi \in \mathcal{E}$, we have

$$\left\| \int_\Gamma (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi dz \right\|_{\mathcal{E}_1} \leqslant \int_\Gamma \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}_1} |dz|,$$

then, applying (8), we obtain

$$\begin{aligned} & \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}_1} \\ & \leqslant C \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}} \\ & + C \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{1-\delta} \left\| \mathcal{M}_1 (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{\delta}. \end{aligned}$$

Now, for all $z \in \Gamma$, we have

$$\begin{aligned} & \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{1-\delta} \left\| \mathcal{M}_1 (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{\delta} \\ & \leqslant \frac{[C_1(\theta_{\mathcal{M}_1}) C_2(\theta_{\mathcal{M}_2})]^{1-\delta}}{|z|^{2(1-\delta)}} \frac{[C_1(\theta_{\mathcal{M}_1})]^{\delta} [C_2(\theta_{\mathcal{M}_2})]^{\delta}}{|z|^{\delta}} \|\xi\|_{\mathcal{E}} = \frac{C_1(\theta_{\mathcal{M}_1}) C_2(\theta_{\mathcal{M}_2})}{|z|^{1+(1-\delta)}} \|\xi\|_{\mathcal{E}}, \end{aligned}$$

from which we deduce the convergence of the integral in (7). □

3 Spectral study of operators

In all the sequel, in view to apply the above results, we will consider the following particular Banach space

$$\mathcal{E} = L^p(0, +\infty; X) \quad \text{with} \quad X = W_0^{2,p}(0, \omega) \times L^p(0, \omega),$$

equipped with its natural norm.

3.1 Study of operator $\mathcal{L}_{1,\mu}$

We study the spectral equation

$$\mathcal{L}_{1,\mu} V - \lambda V = R,$$

where $V \in D(\mathcal{L}_{1,\mu})$, $R \in \mathcal{E}$ and $\lambda \in \mathbb{C}$ (which will be precised below), that is

$$\left\{ \begin{array}{l} V''(t) - 2\mu V'(t) + (\mu^2 - \lambda) V(t) = R(t), \quad t > 0 \\ V(0) = 0, \quad V(+\infty) = 0. \end{array} \right. \tag{9}$$

We set

$$\Pi_\mu = \{ z \in \mathbb{C} \setminus \mathbb{R}_- : \operatorname{Re}(\sqrt{z}) > \mu \}.$$

Now, let us specify Π_μ. For all $z = x + iy \in \mathbb{C} \setminus \mathbb{R}_-$, we have

$$\operatorname{Re}(\sqrt{z}) > \mu \iff \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} > \mu \iff \sqrt{x^2 + y^2} > 2\mu^2 - x.$$

- First case : if $x > \mu^2$, we have $\sqrt{x^2+y^2}+x \geqslant 2x > 2\mu^2$, then $\text{Re}(\sqrt{z}) > \mu$.
- Second case : if $x \leqslant \mu^2$, then $y^2+4\mu^2 x - 4\mu^4 > 0$. Thus, we deduce that Π_μ is strictly outside the parabola of equation

$$y^2 + 4\mu^2 x - 4\mu^4 = 0,$$

turned towards the negative real axis and passing through the points $(\mu^2, 0)$, $(0, 2\mu^2)$ and $(0, -2\mu^2)$.

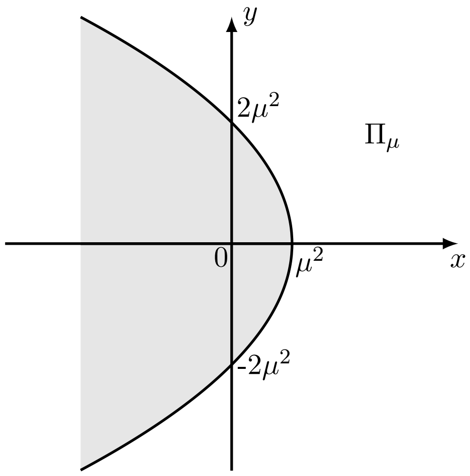

Figure 1: In this figure, Π_μ is the entire uncolored area.

Now, let $\varepsilon_{\mathcal{L}_{1,\mu}}$ be a small fixed positive number and consider the following set

$$\Sigma_{\mathcal{L}_{1,\mu}} := \left\{ \lambda \in \Pi_\mu, \quad |\arg(\lambda)| \leqslant \pi - 2\varepsilon_{\mathcal{L}_{1,\mu}} \quad \text{and} \quad |\lambda| \geqslant \frac{4\mu^2}{\sin^2\left(\varepsilon_{\mathcal{L}_{1,\mu}}\right)} \right\}. \tag{10}$$

We then obtain the following proposition.

Proposition 3.1. The linear operator $\mathcal{L}_{1,\mu}$ is closed and densely defined in $\mathcal{E}$. Moreover, there exists a constant $M_{\mathcal{L}_{1,\mu}} > 0$ such that for all $\lambda \in \Sigma_{\mathcal{L}_{1,\mu}}$, operator $\mathcal{L}_{1,\mu} - \lambda I$ is invertible and

$$\left\| (\mathcal{L}_{1,\mu} - \lambda I)^{-1} \right\|_{\mathcal{L}(\mathcal{E})} \leqslant \frac{M_{\mathcal{L}_{1,\mu}}}{|\lambda|}.$$

Therefore, assumption (H_1) in Section 2.2 is verified for $\mathcal{L}_{1,\mu}$ with

$$\theta_{\mathcal{L}_{1,\mu}} = 2\varepsilon_{\mathcal{L}_{1,\mu}}. \tag{11}$$

Proof. Let $\lambda \in \Pi_\mu$. From Eltaief and Maingot [6], Theorem 2, p. 712, there exists a unique solution $V \in W^{2,p}(0, +\infty; X)$ of problem (9), given by

$$\begin{aligned} V(t) &= \frac{e^{t(\mu-\sqrt{\lambda})}}{2\sqrt{\lambda}} \int_0^{+\infty} e^{-s(\mu+\sqrt{\lambda})} R(s)\, ds \\ &\quad - \frac{1}{2\sqrt{\lambda}} \left(\int_0^t e^{(t-s)(\mu-\sqrt{\lambda})} R(s)\, ds + \int_t^{+\infty} e^{-(s-t)(\mu+\sqrt{\lambda})} R(s)\, ds \right), \end{aligned} \tag{12}$$

see formula (15) in Eltaief and Maingot [6] where $L_1 := -\mu I - \sqrt{\lambda} I$ and $L_2 := \mu I - \sqrt{\lambda} I$. It follows that $\Pi_\mu \subset \rho(\mathcal{L}_{1,\mu})$. This proves that $\mathcal{L}_{1,\mu}$ is closed. The boundary conditions are verified by using Lemma 8, p. 718 in Eltaief and Maingot [6].

Moreover, from (12), we obtain

$$\begin{aligned}\|V\|_{\mathcal{E}} &\leqslant \frac{1}{2\sqrt{|\lambda|}}\left(\int_0^{+\infty} e^{-tp(\mathrm{Re}(\sqrt{\lambda})-\mu)}\,dt\right)^{1/p}\int_0^{+\infty} e^{-s(\mu+\mathrm{Re}(\sqrt{\lambda}))}\|R(s)\|_X\,ds\\ &\quad+\sup_{t\in\mathbb{R}_+}\left(\int_0^t\left|e^{(t-s)(\mu-\sqrt{\lambda})}\right|ds+\int_t^{+\infty}\left|e^{-(s-t)(\mu+\sqrt{\lambda})}\right|ds\right)\frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}},\end{aligned}$$

hence, noting q the conjugate exponent of p, we have

$$\begin{aligned}\|V\|_{\mathcal{E}} &\leqslant \left(\frac{1}{p(\mathrm{Re}(\sqrt{\lambda})-\mu)}\right)^{1/p}\left(\int_0^{+\infty} e^{-sq(\mu+\mathrm{Re}(\sqrt{\lambda}))}\,ds\right)^{1/q}\frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}}\\ &\quad+\sup_{t\in\mathbb{R}_+}\left(\frac{1-e^{-t(\mathrm{Re}(\sqrt{\lambda})-\mu)}}{\mathrm{Re}(\sqrt{\lambda})-\mu}+\frac{1}{\mathrm{Re}(\sqrt{\lambda})+\mu}\right)\frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}}\\ &\leqslant \frac{1}{p^{1/p}(\mathrm{Re}(\sqrt{\lambda})-\mu)^{1/p}}\frac{1}{q^{1/q}(\mathrm{Re}(\sqrt{\lambda})+\mu)^{1/q}}\frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}}\\ &\quad+\frac{2}{\mathrm{Re}(\sqrt{\lambda})-\mu}\frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}}\\ &\leqslant \frac{2}{\mathrm{Re}(\sqrt{\lambda})-\mu}\frac{\|R\|_{\mathcal{E}}}{\sqrt{|\lambda|}}.\end{aligned}$$

Let $\lambda=|\lambda|e^{i\arg(\lambda)}\in\Sigma_{\mathcal{L}_{1,\mu}}$, then $|\arg(\lambda)|\leqslant\pi-2\varepsilon_{\mathcal{L}_{1,\mu}}$. Thus

$$\begin{aligned}\mathrm{Re}(\sqrt{\lambda})-\mu &\geqslant \sqrt{|\lambda|}\cos\left(\frac{\arg(\lambda)}{2}\right)-\mu\geqslant\sqrt{|\lambda|}\cos\left(\frac{\pi}{2}-\varepsilon_{\mathcal{L}_{1,\mu}}\right)-\mu\\ &\geqslant \sqrt{|\lambda|}\sin\left(\varepsilon_{\mathcal{L}_{1,\mu}}\right)-\frac{\sqrt{|\lambda|}}{2}\sin\left(\varepsilon_{\mathcal{L}_{1,\mu}}\right)\geqslant\frac{\sqrt{|\lambda|}}{2}\sin\left(\varepsilon_{\mathcal{L}_{1,\mu}}\right).\end{aligned}$$

We then obtain

$$\|V\|_{\mathcal{E}}\leqslant\frac{M_{\mathcal{L}_{1,\mu}}}{|\lambda|}\|R\|_{\mathcal{E}},\tag{13}$$

where $M_{\mathcal{L}_{1,\mu}}=\frac{4}{\sin\left(\varepsilon_{\mathcal{L}_{1,\mu}}\right)}$. □

3.2 Study of operator $\mathcal{L}_2$

The spectral properties of $\mathcal{L}_2$ are the same as those of its realization $-\mathcal{A}$.

In all this subsection, we assume that

$$\lambda\leqslant 0.$$

We have to solve the following spectral equation in X

$$\mathcal{A}\Psi-\lambda\Psi=F,\tag{14}$$

that is

$$\begin{pmatrix}0 & 1\\ -\left(\dfrac{\partial^2}{\partial\theta^2}+1\right)^2 & -2\left(\dfrac{\partial^2}{\partial\theta^2}-1\right)\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}-\lambda\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}F_1\\ F_2\end{pmatrix},$$

with

$$F_1\in W_0^{2,p}(0,\omega)\quad\text{and}\quad F_2\in L^p(0,\omega).$$

We have to find the unique couple $(\psi_1, \psi_2) \in \left(W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right) \times W_0^{2,p}(0,\omega)$, which satisfies the following system

$$\left\{ \begin{array}{lll} \psi_2 - \lambda\psi_1 & = & F_1 \\ -\psi_1^{(4)} - 2\psi_1'' - \psi_1 - 2\psi_2'' + 2\psi_2 - \lambda\psi_2 & = & F_2. \end{array} \right.$$

Thus, we first have to solve

$$\left\{ \begin{array}{l} -\psi_1^{(4)} - 2\psi_1'' - \psi_1 - 2(\lambda\psi_1 + F_1)'' + (2-\lambda)(\lambda\psi_1 + F_1) = F_2 \\ \psi_1 \in W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega), \end{array} \right.$$

that is

$$\left\{ \begin{array}{l} -\psi_1^{(4)} - 2(1+\lambda)\psi_1'' - (\lambda-1)^2\psi_1 = F_2 + 2F_1'' + (\lambda-2)F_1 \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{array} \right.$$

Set $G_\lambda = -F_2 - 2(F_1'' - F_1) - \lambda F_1$, it follows that the previous system writes

$$\left\{ \begin{array}{l} \psi_1^{(4)} + 2(\lambda+1)\psi_1'' + (\lambda-1)^2\psi_1 = G_\lambda \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{array} \right. \tag{15}$$

Then, the characteristic equation

$$\chi^4 + 2\left(1+\lambda\right)\chi^2 + \left(\lambda-1\right)^2 = 0,$$

admits, for $\lambda < 0$, the following four distinct solutions

$$\left\{ \begin{array}{ll} \alpha_1 = \sqrt{-\lambda} + i, & \alpha_3 = -\alpha_1 \\ \alpha_2 = \sqrt{-\lambda} - i, & \alpha_4 = -\alpha_2, \end{array} \right. \tag{16}$$

and for $\lambda = 0$, two double solutions i and $-i$.

We have to distinguish below two cases : $\lambda = 0$ and $\lambda < 0$.

3.2.1 Case $\lambda = 0$: Invertibility of $\mathcal{A}$

Proposition 3.2. $\mathcal{A}$ is boundedly invertible. Then : $\exists\, \varepsilon_0 > 0 : \overline{B(0,\varepsilon_0)} \subset \rho(\mathcal{A})$.

Proof. Here $\lambda = 0$. We have to solve (14). This is equivalent to solve (15) that is

$$\left\{ \begin{array}{l} \psi_1^{(4)} + 2\psi_1'' + \psi_1 = -F_2 - 2(F_1'' - F_1) \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{array} \right. \tag{17}$$

From Thorel [22], Theorem 2.8, statement 2., there exists a unique classical solution of problem (17) that is $(\psi_1, \psi_2) = (\psi_1, F_1) \in \left(W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right) \times W_0^{2,p}(0,\omega)$. We then deduce that there exists $C_1 > 0$ such that

$$\|\psi_1''\|_{L^p(0,\omega)} \leqslant C_1 \left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{W^{2,p}(0,\omega)}\right) \leqslant 2C_1\|F\|_X,$$

and from the Poincaré inequality, there exists $C_\omega > 0$ such that

$$\|\psi_1\|_{W_0^{2,p}(0,\omega)} \leqslant C_\omega \|\psi_1''\|_{L^p(0,\omega)} \leqslant 2C_1C_\omega\|F\|_X.$$

Finally, since $\psi_2 = F_1$, we have

$$\|\psi_2\|_{L^p(0,\omega)} = \|F_1\|_{L^p(0,\omega)} \leqslant \|F_2\|_{L^p(0,\omega)} + \|F_1\|_{W^{2,p}(0,\omega)} = \|F\|_X.$$

□

3.2.2 Case $\lambda < 0$: Spectral study of $\mathcal{A}$

In order to prove Proposition 3.9, we first have to state the following technical results.

Lemma 3.3. Let $\alpha \in \mathbb{C} \setminus \{0\}$, $a, b \in \mathbb{R}$ with $a < b$ and $f \in W_0^{2,p}(a,b)$. For all $x \in [a,b]$, we set

$$K(x) = \int_a^x e^{-(x-s)\alpha} f(s)\, ds + \int_x^b e^{-(s-x)\alpha} f(s)\, ds.$$

Then, we have

$$K(x) = \frac{2}{\alpha} f(x) + \frac{1}{\alpha^2} \int_a^x e^{-(x-s)\alpha} f''(s)\, ds + \frac{1}{\alpha^2} \int_x^b e^{-(s-x)\alpha} f''(s)\, ds.$$

Proof. The result is easily obtained by two integrations by parts. □

Now, let us solve explicitly problem (15) for $\lambda < 0$.

Proposition 3.4. Problem (15) has a unique solution which can be written in the following form

$$\begin{aligned} \psi_1(\theta) \;:=\; & e^{-\theta\alpha_2}(\beta_1+\beta_2+\beta_3+\beta_4) + e^{-(\omega-\theta)\alpha_2}(\beta_3+\beta_4-\beta_1-\beta_2) + S(\theta) \\ & + \left(e^{-\theta\alpha_1} - e^{-\theta\alpha_2}\right)(\beta_2+\beta_4) + \left(e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2}\right)(\beta_4-\beta_2), \end{aligned} \tag{18}$$

where the constants β_i, $i = 1,2,3,4$, and the particular solution S will be explicitly given below.

Proof. In order to apply results obtained in Labbas, Lemrabet, Maingot and Thorel [12] and Labbas, Maingot, Manceau and Thorel [13], we set

$$L_- = -\alpha_1 I, \quad M = -\alpha_2 I, \quad r_- = \alpha_1^2 - \alpha_2^2, \quad a = 0 \quad \text{and} \quad b = \omega.$$

Then, problem (15) can be written as

$$\left\{ \begin{array}{l} \psi_1^{(4)} - (L_-^2 + M^2)\psi_1'' + L_-^2 M^2 \psi_1 = G_\lambda \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{array} \right. \tag{19}$$

From Labbas, Maingot, Manceau and Thorel [13], there exists a unique classical solution ψ_1 of problem (19). Its representation formula, in the form (18), is explicitly given in Labbas, Lemrabet, Maingot and Thorel [12] by (14)-(15)-(16) or also in a more general framework in Labbas, Maingot and Thorel [15] by (22)-(30)-(31).

For the reader convenience, we will prove, by a long calculus, that

$$\left\{ \begin{array}{lcl} \beta_1 & = & \dfrac{1}{4i} U_1^{-1} \dfrac{1 - e^{-\omega\alpha_1}}{1 - e^{-\omega\alpha_2}} \left(J(0) - J(\omega)\right) \\ \beta_2 & = & -\dfrac{1}{4i} U_1^{-1} \left(J(0) - J(\omega)\right) \\ \beta_3 & = & -\dfrac{1}{4i} U_2^{-1} \dfrac{1 + e^{-\omega\alpha_1}}{1 + e^{-\omega\alpha_2}} \left(J(0) + J(\omega)\right) \\ \beta_4 & = & \dfrac{1}{4i} U_2^{-1} \left(J(0) + J(\omega)\right), \end{array} \right. \tag{20}$$

with

$$\left\{ \begin{array}{lcl} U_1 & := & 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \\ U_2 & := & 1 - e^{-2\omega\sqrt{-\lambda}} + 2\omega\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}}, \end{array} \right. \tag{21}$$

and for all $\theta \in [0,\omega]$

$$\begin{aligned} S(\theta) &:= \frac{e^{-\theta\alpha_2}}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(0)-e^{-\omega\alpha_2}J(\omega)\right)-\frac{\lambda}{\alpha_1^2\alpha_2^2}F_1(\theta) \\ &\quad +\frac{e^{-(\omega-\theta)\alpha_2}}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(\omega)-e^{-\omega\alpha_2}J(0)\right)-\frac{1}{2\alpha_2}J(\theta), \end{aligned} \tag{22}$$

with

$$J(\theta) := \int_0^\theta e^{-(\theta-s)\alpha_2}v(s)\;ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2}v(s)\;ds, \tag{23}$$

where

$$\begin{aligned} v(\theta) &:= \frac{e^{-\theta\alpha_1}}{2\alpha_1\left(1-e^{-2\omega\alpha_1}\right)}\left(I(0)-e^{-\omega\alpha_1}I(\omega)\right)+\frac{\lambda}{\alpha_1^2\alpha_2^2}F_1''(\theta) \\ &\quad +\frac{e^{-(\omega-\theta)\alpha_1}}{2\alpha_1\left(1-e^{-2\omega\alpha_1}\right)}\left(I(\omega)-e^{-\omega\alpha_1}I(0)\right)-\frac{1}{2\alpha_1}I(\theta), \end{aligned} \tag{24}$$

and

$$\begin{aligned} I(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1}\left(-F_2-2(F_1''-F_1)+\frac{\lambda}{\alpha_1^2}F_1''\right)(s)\;ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1}\left(-F_2-2(F_1''-F_1)+\frac{\lambda}{\alpha_1^2}F_1''\right)(s)\;ds. \end{aligned} \tag{25}$$

Let us begin our proof. Here, we are inspired by the formulas given by (15) and (16), p. 2943, in Labbas, Lemrabet, Maingot and Thorel [12], where the authors have used the notations F_-, f_-, U_- and V_- which are replace respectively here by S, G_λ, U_1 and U_2. Consequently, we have

$$\begin{aligned} S(\theta) &= \frac{Ze^{-\theta\alpha_2}}{2\alpha_2}\int_0^\omega e^{-s\alpha_2}v_0(s)\;ds + \frac{Ze^{-(\omega-\theta)\alpha_2}}{2\alpha_2}\int_0^\omega e^{-(\omega-s)\alpha_2}v_0(s)\;ds \\ &\quad -\frac{1}{2\alpha_2}\int_0^\theta e^{-(\theta-s)\alpha_2}v_0(s)\;ds - \frac{1}{2\alpha_2}\int_\theta^\omega e^{-(s-\theta)\alpha_2}v_0(s)\;ds \\ &\quad -\frac{Ze^{-\omega\alpha_2}e^{-\theta\alpha_2}}{2\alpha_2}\int_0^\omega e^{-(\omega-s)\alpha_2}v_0(s)\;ds \\ &\quad -\frac{Ze^{-\omega\alpha_2}e^{-(\omega-\theta)\alpha_2}}{2\alpha_2}\int_0^\omega e^{-s\alpha_2}v_0(s)\;ds, \end{aligned} \tag{26}$$

where

$$\begin{aligned} v_0(\theta) &:= \frac{We^{-\theta\alpha_1}}{2\alpha_1}\int_0^\omega e^{-s\alpha_1}G_\lambda(s)\;ds + \frac{We^{-(\omega-\theta)\alpha_1}}{2\alpha_1}\int_0^\omega e^{-(\omega-s)\alpha_1}G_\lambda(s)\;ds \\ &\quad -\frac{We^{-\omega\alpha_1}e^{-\theta\alpha_1}}{2\alpha_1}\int_0^\omega e^{-(\omega-s)\alpha_1}G_\lambda(s)\;ds \\ &\quad -\frac{We^{-\omega\alpha_1}e^{-(\omega-\theta)\alpha_1}}{2\alpha_1}\int_0^\omega e^{-s\alpha_1}G_\lambda(s)\;ds - \frac{1}{2\alpha_1}I_1(\theta), \end{aligned} \tag{27}$$

with $Z := \left(1-e^{-2\omega\alpha_2}\right)^{-1}$, $W := \left(1-e^{-2\omega\alpha_1}\right)^{-1}$ and

$$I_1(\theta) = \int_0^\theta e^{-(\theta-s)\alpha_1}G_\lambda(s)\;ds + \int_\theta^\omega e^{-(s-\theta)\alpha_1}G_\lambda(s)\;ds. \tag{28}$$

Then, since $G_\lambda = -F_2 - 2(F_1''-F_1) - \lambda F_1$, we have

$$\begin{aligned} I_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1}\left(-F_2-2(F_1''-F_1)\right)(s)\;ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1}\left(-F_2-2(F_1''-F_1)\right)(s)\;ds \\ &\quad -\lambda\int_0^\theta e^{-(\theta-s)\alpha_1}F_1(s)\;ds - \lambda\int_\theta^\omega e^{-(s-\theta)\alpha_1}F_1(s)\;ds. \end{aligned}$$

From Lemma 3.3 and the fact that $F_1 \in W_0^{2,p}(0,\omega)$, it follows that

$$\begin{aligned}
I_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1}\left(-F_2 - 2(F_1''-F_1)\right)(s)\,ds \\
&\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1}\left(-F_2 - 2(F_1''-F_1)\right)(s)\,ds \\
&\quad - \frac{2\lambda}{\alpha_1}F_1(\theta) + \frac{\lambda}{\alpha_1^2}\left(\int_0^\theta e^{-(\theta-s)\alpha_1}F_1''(s)\,ds + \int_\theta^\omega e^{-(s-\theta)\alpha_1}F_1''(s)\,ds\right) \\
&= -\frac{2\lambda}{\alpha_1}F_1(\theta) + \int_0^\theta e^{-(\theta-s)\alpha_1}\left(-F_2 - 2(F_1''-F_1) + \frac{\lambda}{\alpha_1^2}F_1''\right)(s)\,ds \\
&\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1}\left(-F_2 - 2(F_1''-F_1) + \frac{\lambda}{\alpha_1^2}F_1''\right)(s)\,ds.
\end{aligned}$$

Thus I, given by (25), satisfies

$$I(\theta) = I_1(\theta) + \frac{2\lambda}{\alpha_1}F_1(\theta). \tag{29}$$

Note that, from (28) and (29), we have

$$\int_0^\omega e^{-s\alpha_1}G_\lambda(s)\,ds = I_1(0) = I(0) \quad \text{and} \quad \int_0^\omega e^{-(\omega-s)\alpha_1}G_\lambda(s)\,ds = I_1(\omega) = I(\omega).$$

Therefore, from (27), we deduce that for all $\theta \in [0,\omega]$

$$\begin{aligned}
v_0(\theta) &= \frac{We^{-\theta\alpha_1}}{2\alpha_1}\left(I(0) - e^{-\omega\alpha_1}I(\omega)\right) \\
&\quad + \frac{We^{-(\omega-\theta)\alpha_1}}{2\alpha_1}\left(I(\omega) - e^{-\omega\alpha_1}I(0)\right) - \frac{1}{2\alpha_1}I_1(\theta) \\
&= \frac{We^{-\theta\alpha_1}}{2\alpha_1}\left(I(0) - e^{-\omega\alpha_1}I(\omega)\right) + \frac{\lambda}{\alpha_1^2}F_1(\theta) \\
&\quad + \frac{We^{-(\omega-\theta)\alpha_1}}{2\alpha_1}\left(I(\omega) - e^{-\omega\alpha_1}I(0)\right) - \frac{1}{2\alpha_1}I(\theta).
\end{aligned}$$

Set

$$v_1 = v_0 - \frac{\lambda}{\alpha_1^2}F_1, \tag{30}$$

and

$$J_1(\theta) = \int_0^\theta e^{-(\theta-s)\alpha_2}v_0(s)\,ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2}v_0(s)\,ds; \tag{31}$$

then, due to Lemma 3.3, we obtain

$$\begin{aligned}
J_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_2}v_1(s)\,ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2}v_1(s)\,ds \\
&\quad + \frac{\lambda}{\alpha_1^2}\left(\int_0^\theta e^{-(\theta-s)\alpha_2}F_1(s)\,ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2}F_1(s)\,ds\right) \\
&= \int_0^\theta e^{-(\theta-s)\alpha_2}\left(v_1(s) + \frac{\lambda}{\alpha_1^2\alpha_2^2}F_1''(s)\right)ds \\
&\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_2}\left(v_1(s) + \frac{\lambda}{\alpha_1^2\alpha_2^2}F_1''(s)\right)ds + \frac{2\lambda}{\alpha_1^2\alpha_2}F_1(\theta).
\end{aligned}$$

From (30), for all $\theta \in [0, \omega]$, we deduce that v given by (24) and J given by (23), satisfy

$$v(\theta) = v_1(\theta) + \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1''(\theta),$$

and

$$J(\theta) = J_1(\theta) - \frac{2\lambda}{\alpha_1^2 \alpha_2} F_1(\theta). \tag{32}$$

Note that, from (31) and (32), we have

$$\int_0^\omega e^{-s\alpha_2} v_0(s)\, ds = J_1(0) = J(0) \quad \text{and} \quad \int_0^\omega e^{-(\omega - s)\alpha_2} v_0(s)\, ds = J_1(\omega) = J(\omega).$$

Finally, from (23), (26), (32) and for all $\theta \in [0, \omega]$, we deduce that

$$\begin{aligned} S(\theta) &= \frac{Z e^{-\theta \alpha_2}}{2\alpha_2} \left(J(0) - e^{-\omega \alpha_2} J(\omega) \right) \\ &\quad + \frac{Z e^{-(\omega - \theta)\alpha_2}}{2\alpha_2} \left(J(\omega) - e^{-\omega \alpha_2} J(0) \right) - \frac{1}{2\alpha_2} J_1(\theta), \end{aligned}$$

which is (22).

Now, in order to compute β_i, $i = 1, ..., 4$, we must explicit U_1, U_2 and $S'(0) \pm S'(\omega)$.

$$\begin{aligned} U_1 &= 1 - e^{-\omega(\alpha_1 + \alpha_2)} - (\alpha_1^2 - \alpha_2^2)^{-1} (\alpha_1 + \alpha_2)^2 \left(e^{-\omega \alpha_2} - e^{-\omega \alpha_1} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} + i\sqrt{-\lambda} \left(e^{-\omega(\sqrt{-\lambda} - i)} - e^{-\omega(\sqrt{-\lambda} + i)} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} - 2\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \sin(\omega), \end{aligned}$$

and

$$\begin{aligned} U_2 &= 1 - e^{-\omega(\alpha_1 + \alpha_2)} + (\alpha_1^2 - \alpha_2^2)^{-1} (\alpha_1 + \alpha_2)^2 \left(e^{-\omega \alpha_2} - e^{-\omega \alpha_1} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} - i\sqrt{-\lambda} \left(e^{-\omega(\sqrt{-\lambda} - i)} - e^{-\omega(\sqrt{-\lambda} + i)} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} + 2\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \sin(\omega). \end{aligned}$$

From (22), it follows that

$$\begin{aligned} S'(\theta) &= -\frac{Z e^{-\theta \alpha_2}}{2} \left(J(0) - e^{-\omega \alpha_2} J(\omega) \right) - \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1'(\theta) \\ &\quad + \frac{Z e^{-(\omega - \theta)\alpha_2}}{2} \left(J(\omega) - e^{-\omega \alpha_2} J(0) \right) - \frac{1}{2\alpha_2} J'(\theta) \\ &= -\frac{Z e^{-\theta \alpha_2}}{2} \left(J(0) - e^{-\omega \alpha_2} J(\omega) \right) + \frac{Z e^{-(\omega - \theta)\alpha_2}}{2} \left(J(\omega) - e^{-\omega \alpha_2} J(0) \right) \\ &\quad + \frac{1}{2} \left(\int_0^\theta e^{-(\theta - s)\alpha_2} v(s)\, ds - \int_\theta^\omega e^{-(s - \theta)\alpha_2} v(s)\, ds \right) - \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1'(\theta), \end{aligned}$$

then

$$S'(0) + S'(\omega) = -\frac{J(0) - J(\omega)}{(1 - e^{-\omega \alpha_2})} \quad \text{and} \quad S'(0) - S'(\omega) = -\frac{J(0) + J(\omega)}{(1 + e^{-\omega \alpha_2})}.$$

from which we deduce the constants β_i, $i = 1, 2, 3, 4$, written in (20). □

Remark 3.5. Since $0 < \sin(\omega) < \omega$, for all $\omega > 0$, then we have

$$U_1 = 1 - e^{-2\omega\sqrt{-\lambda}} - 2\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \sin(\omega) \geqslant 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}},$$

and

$$U_2 = 1 - e^{-2\omega\sqrt{-\lambda}} + 2\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \sin(\omega) \geqslant 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}}.$$

For $x > 0$, we consider the following function

$$h(x) = 1 - e^{-2x} - 2xe^{-x};$$

we then have

$$h'(x) = 2e^{-2x} - 2e^{-x} + 2xe^{-x} = 2e^{-x}\left(e^{-x} + x - 1\right) > 0.$$

It follows that $h(x) > h(0) = 0$, for all $x > 0$. Finally, we deduce that

$$U_1 \geqslant 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}} = h(\omega\sqrt{-\lambda}) > 0, \tag{33}$$

and

$$U_2 \geqslant 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}} = h(\omega\sqrt{-\lambda}) > 0. \tag{34}$$

Lemma 3.6. For all $\lambda \in (-\infty, -\varepsilon_0]$, where ε_0 is defined in Proposition 3.2, functions J, v and I, given by (23), (24) and (25), satisfy the following estimates

1. $\|I\|_{L^p(0,\omega)} \leqslant \dfrac{2}{\sqrt{-\lambda}} \Big(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\Big)$.

2. $|I(0)| + |I(\omega)| \leqslant \dfrac{2}{\sqrt{-\lambda}^{1-1/p}} \left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)$.

3. $\|v\|_{L^p(0,\omega)} \leqslant \dfrac{M_1}{-\lambda} \Big(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\Big)$,

   where $M_1 = 2 + \frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}$.

4. $\|J\|_{L^p(0,\omega)} \leqslant \dfrac{2}{\sqrt{-\lambda}} \|v\|_{L^p(0,\omega)}$.

5. $|J(0)| + |J(\omega)| \leqslant \dfrac{2}{\sqrt{-\lambda}^{1-1/p}} \|v\|_{L^p(0,\omega)}$.

Proof.

1. From (16) and (25), we obtain

$$\begin{aligned}
\|I\|_{L^p(0,\omega)} &\leqslant \sup_{\theta\in[0,\omega]} \left(\int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}}\, ds + \int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}}\, ds\right) \|2F_1 - F_2\|_{L^p(0,\omega)} \\
&\quad + \sup_{\theta\in[0,\omega]} \left(\int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}}\, ds + \int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}}\, ds\right) \left\|\left(\frac{\lambda}{\alpha_1^2} - 2\right) F_1''\right\|_{L^p(0,\omega)} \\
&\leqslant \sup_{\theta\in[0,\omega]} \left(\frac{1 - e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}} + \frac{1 - e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}}\right) \Big(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)}\Big) \\
&\quad + 3 \sup_{\theta\in[0,\omega]} \left(\frac{1 - e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}} + \frac{1 - e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}}\right) \|F_1''\|_{L^p(0,\omega)} \\
&\leqslant \frac{2}{\sqrt{-\lambda}} \Big(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\Big).
\end{aligned}$$

2. Due to (16), (25) and the Hölder inequality, it follows

$$\begin{aligned}
|I(0)|+|I(\omega)| &\leqslant \int_0^\omega e^{-s\sqrt{-\lambda}}\left|-F_2(s)-2(F_1''(s)-F_1(s))+\frac{\lambda}{\alpha_1^2}F_1''(s)\right|ds\\
&\quad+\int_0^\omega e^{-(\omega-s)\sqrt{-\lambda}}\left|-F_2(s)-2(F_1''(s)-F_1(s))+\frac{\lambda}{\alpha_1^2}F_1''(s)\right|ds\\
&\leqslant \left(\int_0^\omega e^{-q(\omega-s)\sqrt{-\lambda}}\,ds\right)^{1/q}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\quad+\left(\int_0^\omega e^{-sq\sqrt{-\lambda}}\,ds\right)^{1/q}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\leqslant \frac{2\left(1-e^{-\omega q\sqrt{-\lambda}}\right)^{1/q}}{q^{1/q}\sqrt{-\lambda}^{1/q}}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\leqslant \frac{2}{\sqrt{-\lambda}^{1-1/p}}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right).
\end{aligned}$$

3. From (16), (24) and the fact that $|\alpha_1|=|\alpha_2|=\sqrt{1-\lambda}>\sqrt{-\lambda}$, we have

$$\begin{aligned}
\|v\|_{L^p(0,\omega)} &\leqslant \frac{|I(0)|+|I(\omega)|}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\,d\theta\right)^{1/p}+\frac{-\lambda}{(1-\lambda)^2}\|F_1''\|_{L^p(0,\omega)}\\
&\quad+\frac{|I(0)|+|I(\omega)|}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\,d\theta\right)^{1/p}+\frac{1}{2\sqrt{1-\lambda}}\|I\|_{L^p(0,\omega)}\\
&\leqslant \frac{(|I(0)|+|I(\omega)|)}{\sqrt{1-\lambda}\sqrt{-\lambda}^{1/p}p^{1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}+\frac{1}{2\sqrt{1-\lambda}}\|I\|_{L^p(0,\omega)}\\
&\quad+\frac{1}{1-\lambda}\|F_1''\|_{L^p(0,\omega)},
\end{aligned}$$

hence

$$\begin{aligned}
\|v\|_{L^p(0,\omega)} &\leqslant \frac{2\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{\sqrt{1-\lambda}\sqrt{-\lambda}^{1/p+1/q}\,q^{1/q}\,p^{1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\\
&\quad+\frac{\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}}{\sqrt{1-\lambda}\sqrt{-\lambda}}+\frac{1}{1-\lambda}\|F_1''\|_{L^p(0,\omega)}\\
&\leqslant \frac{M_1}{\sqrt{1-\lambda}\sqrt{-\lambda}}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right).
\end{aligned}$$

4. From (16) and (23), we have

$$\begin{aligned}
\|J\|_{L^p(0,\omega)} &\leqslant \sup_{\theta\in[0,\omega]}\left(\int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}}\,ds+\int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}}\,ds\right)\|v\|_{L^p(0,\omega)}\\
&\leqslant \sup_{\theta\in[0,\omega]}\left(\frac{1-e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}}+\frac{1-e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}}\right)\|v\|_{L^p(0,\omega)}\\
&\leqslant \frac{2}{\sqrt{-\lambda}}\|v\|_{L^p(0,\omega)}.
\end{aligned}$$

5. Due to (16), (23) and the Hölder inequality, we deduce that

$$\begin{aligned}
|J(0)|+|J(\omega)| &\leqslant \int_0^\omega e^{-s\sqrt{-\lambda}}|v(s)|\,ds+\int_0^\omega e^{-(\omega-s)\sqrt{-\lambda}}|v(s)|\,ds\\
&\leqslant \left(\left(\int_0^\omega e^{-sq\sqrt{-\lambda}}\,ds\right)^{1/q}+\left(\int_0^\omega e^{-(\omega-s)q\sqrt{-\lambda}}\,ds\right)^{1/q}\right)\|v\|_{L^p(0,\omega)}\\
&\leqslant \frac{2\left(1-e^{-\omega q\sqrt{-\lambda}}\right)^{1/q}}{q^{1/q}\sqrt{-\lambda}^{1/q}}\|v\|_{L^p(0,\omega)}\\
&\leqslant \frac{2}{\sqrt{-\lambda}^{1-1/p}}\|v\|_{L^p(0,\omega)}.
\end{aligned}$$

□

Lemma 3.7. Let $\lambda<0$. Then, we have

$$\left(\int_0^\omega \left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\,d\theta\right)^{1/p}\leqslant\frac{4}{\sqrt{-\lambda}^{1+1/p}},$$

and

$$\left(\int_0^\omega \left|e^{-(\omega-\theta)\alpha_1}-e^{-(\omega-\theta)\alpha_2}\right|^p\,d\theta\right)^{1/p}\leqslant\frac{4}{\sqrt{-\lambda}^{1+1/p}}.$$

Proof. For $x\geqslant 0$, we have $e^{-\frac{px}{2}}x^p<1$, so

$$\int_0^{+\infty}e^{-px}x^p\,dx=\int_0^{+\infty}e^{-\frac{px}{2}}e^{-\frac{px}{2}}x^p\,dx\leqslant\int_0^{+\infty}e^{-\frac{px}{2}}\,dx=\frac{2}{p}.$$

Then, from (16), we have

$$\int_0^\omega\left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\,d\theta=\int_0^\omega\left|e^{-\theta\sqrt{-\lambda}}\left(e^{-\theta i}-e^{\theta i}\right)\right|^p\,d\theta=2^p\int_0^\omega e^{-p\theta\sqrt{-\lambda}}|\sin(\theta)|^p\,d\theta,$$

hence, setting $x=\theta\sqrt{-\lambda}$, it follows that

$$\begin{aligned}
\int_0^\omega\left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\,d\theta &= 2^p\int_0^{\omega\sqrt{-\lambda}}e^{-px}\left|\sin\left(\frac{x}{\sqrt{-\lambda}}\right)\right|^p\,\frac{dx}{\sqrt{-\lambda}}\\
&\leqslant \frac{2^p}{\sqrt{-\lambda}}\int_0^{\omega\sqrt{-\lambda}}e^{-px}\left(\frac{x}{\sqrt{-\lambda}}\right)^p\,dx\\
&\leqslant \frac{2^p}{\sqrt{-\lambda}^{p+1}}\int_0^{+\infty}e^{-px}x^p\,dx\\
&\leqslant \frac{2^{p+1}}{p\sqrt{-\lambda}^{p+1}}\leqslant\frac{2^{2p}}{\sqrt{-\lambda}^{p+1}}.
\end{aligned}$$

The second estimate is obtained by change of variable, taking $\omega-\theta$ instead of θ. □

Lemma 3.8. For all $\lambda\in(-\infty,-\varepsilon_0]$, where ε_0 is defined in Proposition 3.2, the constants β_1, β_2, β_3 and β_4, defined by (20), satisfy

$$\max\left(|\beta_1+\beta_2|,|\beta_3+\beta_4|\right)\leqslant\frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1-e^{-\omega\sqrt{\varepsilon_0}}\right)},$$

and

$$\max\left(|\beta_2|,|\beta_4|\right) \leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)},$$

where $M_1 = 2+\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}$.

Proof. Recall that β_i, $i=1,2,3,4$, depends on U_1^{-1} and U_2^{-1}. From (33) and (34), it follows

$$U_1 \geqslant h(\omega\sqrt{-\lambda}) \geqslant h(\omega\sqrt{\varepsilon_0}) = 1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}} > 0,$$

and

$$U_2 \geqslant h(\omega\sqrt{-\lambda}) \geqslant h(\omega\sqrt{\varepsilon_0}) = 1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}} > 0.$$

Thus, we deduce

$$U_1^{-1} \leqslant \frac{1}{1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}} \quad \text{and} \quad U_2^{-1} \leqslant \frac{1}{1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}}.$$

Therefore, from (20) and Lemma 3.6, we have

$$\begin{aligned}
|\beta_1+\beta_2| &\leqslant \frac{|J(0)|+|J(\omega)|}{4\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)}\left|\frac{1-e^{-\omega\alpha_1}}{1-e^{-\omega\alpha_2}}-1\right| \\
&\leqslant \frac{\|v\|_{L^p(0,\omega)}}{2\sqrt{-\lambda}^{1-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)}\left|\frac{e^{-\omega\alpha_2}-e^{-\omega\alpha_1}}{1-e^{-\omega\alpha_2}}\right| \\
&\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)}\,\frac{2e^{-\omega\sqrt{-\lambda}}}{1-e^{-\omega\sqrt{\varepsilon_0}}} \\
&\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1-e^{-\omega\sqrt{\varepsilon_0}}\right)}
\end{aligned}$$

and similarly

$$\begin{aligned}
|\beta_3+\beta_4| &\leqslant \frac{|J(0)|+|J(\omega)|}{4\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)}\left|1-\frac{1+e^{-\omega\alpha_1}}{1+e^{-\omega\alpha_2}}\right| \\
&\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1-e^{-\omega\sqrt{\varepsilon_0}}\right)}.
\end{aligned}$$

In the same way, we obtain

$$|\beta_2| \leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)},$$

and

$$|\beta_4| \leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)}.$$

□

Proposition 3.9. $\mathcal{A}$ is closed and densely defined in X. Moreover, there exists a constant $M > 0$ such that for all $\lambda \leqslant 0$, operator $\mathcal{A} - \lambda I$ is invertible with bounded inverse and

$$\left\|(\mathcal{A} - \lambda I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{M}{1+|\lambda|}.$$

Remark 3.10. This result implies, in particular, that $-\sqrt{\mathcal{A}}$ is well-defined and generates a uniformly bounded analytic semigroup $\left(e^{-s\sqrt{\mathcal{A}}}\right)_{s\geqslant 0}$.

Proof. Recall that

$$D(\mathcal{A}) = \left[W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right] \times W_0^{2,p}(0,\omega).$$

It is clear that

$$\mathcal{D}(0,\omega) \times \mathcal{D}(0,\omega) \subset D(\mathcal{A}) \subset X = W_0^{2,p}(0,\omega) \times L^p(0,\omega),$$

where $\mathcal{D}(0,\omega)$ is the set of C^∞-functions with compact support in $(0,\omega)$. Since $\mathcal{D}(0,\omega)$ is dense in each spaces $W_0^{2,p}(0,\omega)$ and $L^p(0,\omega)$ for their respective norms, then $D(\mathcal{A})$ is dense in X.

From Proposition 3.2, $0 \in \rho(\mathcal{A})$, thus $\mathcal{A}$ is closed. From Proposition 3.4, for all $\lambda < 0$, there exists a unique couple

$$(\psi_1, \psi_2) \in \left(W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right) \times W_0^{2,p}(0,\omega)$$

which satisfies

$$\left\{\begin{array}{lcl} \psi_2 & = & \lambda\psi_1 + F_1 \\ \psi_1^{(4)} + 2(\lambda+1)\psi_1'' + (\lambda-1)^2\psi_1 & = & G_\lambda, \end{array}\right. \tag{35}$$

where $G_\lambda = -F_2 - 2(F_1'' - F_1) - \lambda F_1$. Then $\mathbb{R}_- \subset \rho(\mathcal{A})$ and

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (\mathcal{A} - \lambda I)^{-1} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = (\mathcal{A} - \lambda I)^{-1} F,$$

where ψ_1 is given by (18)-(20)-(22) and

$$\begin{array}{rcl} \psi_2(\theta) & := & \lambda e^{-\theta\alpha_2}(\beta_1+\beta_2+\beta_3+\beta_4) + \lambda e^{-(\omega-\theta)\alpha_2}(\beta_3+\beta_4-\beta_1-\beta_2) \\ & & +\lambda\left(e^{-\theta\alpha_1} - e^{-\theta\alpha_2}\right)(\beta_2+\beta_4) + \lambda\left(e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2}\right)(\beta_4-\beta_2) \\ & & +\lambda S(\theta) + F_1(\theta), \end{array} \tag{36}$$

with β_i, $i = 1,2,3,4$ are defined in (20)-(21). From (22), we have

$$\begin{array}{rcl} \lambda S(\theta) + F_1(\theta) & = & \dfrac{\lambda}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}\, e^{-\theta\alpha_2}\left(J(0) - e^{-\omega\alpha_2}J(\omega)\right) \\ & & +\dfrac{\lambda}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}\, e^{-(\omega-\theta)\alpha_2}\left(J(\omega) - e^{-\omega\alpha_2}J(0)\right) \\ & & -\dfrac{\lambda^2}{\alpha_1^2\alpha_2^2}\, F_1(\theta) + F_1(\theta) - \dfrac{\lambda}{2\alpha_2}\, J(\theta), \end{array} \tag{37}$$

where $J(\theta)$ is given by (23). Our aim is to prove that, for all $\lambda \leqslant 0$, there exists $M > 0$, such that

$$\|(\mathcal{A} - \lambda I)^{-1}F\|_{\mathcal{L}(X)} \leqslant \frac{M}{1+|\lambda|}\|F\|_X,$$

with

$$\|F\|_X = \left\|\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\right\|_X = \|F_1\|_{W_0^{2,p}(0,\omega)} + \|F_2\|_{L^p(0,\omega)}. \tag{38}$$

To this end, we consider that $\lambda \in (-\infty, -\varepsilon_0]$, where ε_0 is defined in Proposition 3.2. We first study S'' and ψ_1''. From (22), for *a.e.* $\theta \in [0, \omega]$, we have

$$\begin{aligned} S''(\theta) &= \frac{\alpha_2\, e^{-\theta\alpha_2}}{2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(0) - e^{-\omega\alpha_2}J(\omega)\right) - \frac{\lambda}{\alpha_1^2\alpha_2^2}F_1''(\theta) \\ &\quad + \frac{\alpha_2\, e^{-(\omega-\theta)\alpha_2}}{2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(\omega) - e^{-\omega\alpha_2}J(0)\right) - \frac{1}{2\alpha_2}J''(\theta), \end{aligned}$$

and from (23), we obtain $J''(\theta) = \alpha_2^2 J(\theta) - 2\alpha_2 v(\theta)$, hence

$$\begin{aligned} S''(\theta) &= \frac{\alpha_2\, e^{-\theta\alpha_2}}{2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(0) - e^{-\omega\alpha_2}J(\omega)\right) - \frac{\lambda}{\alpha_1^2\alpha_2^2}F_1''(\theta) \\ &\quad + \frac{\alpha_2\, e^{-(\omega-\theta)\alpha_2}}{2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(\omega) - e^{-\omega\alpha_2}J(0)\right) - \frac{\alpha_2}{2}\, J(\theta) + v(\theta). \end{aligned}$$

Then, since $\alpha_1 = \sqrt{-\lambda} + i$ and $\alpha_2 = \sqrt{-\lambda} - i$, we have $|e^{-\omega\alpha_1}| = |e^{-\omega\alpha_2}| = e^{-\omega\sqrt{-\lambda}} \leqslant 1$ with $-\lambda \geqslant \varepsilon_0$, thus

$$\begin{aligned} \|S''\|_{L^p(0,\omega)} &\leqslant \frac{\sqrt{1-\lambda}\,\left(|J(0)| + |J(\omega)|\right)}{2\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\, d\theta\right)^{1/p} \\ &\quad + \frac{\sqrt{1-\lambda}\,\left(|J(0)| + |J(\omega)|\right)}{2\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\, d\theta\right)^{1/p} \\ &\quad + \frac{-\lambda}{(1-\lambda)^2}\, \|F_1''\|_{L^p(0,\omega)} + \frac{\sqrt{1-\lambda}}{2}\, \|J\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} \\ &\leqslant \frac{\sqrt{1-\lambda}\,\left(|J(0)| + |J(\omega)|\right)}{\sqrt{-\lambda}^{1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)} + \frac{1}{1-\lambda}\, \|F_1''\|_{L^p(0,\omega)} \\ &\quad + \frac{\sqrt{1-\lambda}}{2}\, \|J\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)}. \end{aligned}$$

From Lemma 3.6, we have

$$\begin{aligned} \|S''\|_{L^p(0,\omega)} &\leqslant \frac{2\sqrt{1-\lambda}}{\sqrt{-\lambda}^{1-1/p+1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\, \|v\|_{L^p(0,\omega)} \\ &\quad + \frac{\sqrt{1-\lambda}}{\sqrt{-\lambda}}\, \|v\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} + \frac{1}{1-\lambda}\, \|F_1''\|_{L^p(0,\omega)} \\ &\leqslant \frac{2M_1}{-\lambda\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right) \\ &\quad + \frac{2M_1}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right) + \frac{1}{1-\lambda}\, \|F_1''\|_{L^p(0,\omega)}. \end{aligned}$$

Finally, we obtain

$$\|S''\|_{L^p(0,\omega)} \leqslant \frac{M_2}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right), \tag{39}$$

where $M_2 = \frac{2M_1}{1-e^{-2\omega\sqrt{\varepsilon_0}}} + 2M_1 + 1$.

Now, we have

$$\begin{aligned}\psi_1''(\theta)-S''(\theta) &= \alpha_2^2\, e^{-\theta\alpha_2}(\beta_1+\beta_2+\beta_3+\beta_4)+\alpha_2^2\, e^{-(\omega-\theta)\alpha_2}(\beta_3+\beta_4-\beta_1-\beta_2)\\ &\quad +\alpha_2^2\left(e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right)(\beta_2+\beta_4)+\alpha_2^2\left(e^{-(\omega-\theta)\alpha_1}-e^{-(\omega-\theta)\alpha_2}\right)(\beta_4-\beta_2).\end{aligned}$$

Then

$$\begin{aligned}\|\psi_1''-S''\|_{L^p(0,\omega)} &\leqslant (1-\lambda)(|\beta_1+\beta_2|+|\beta_3+\beta_4|)\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\, d\theta\right)^{1/p}\\ &\quad +(1-\lambda)(|\beta_1+\beta_2|+|\beta_3+\beta_4|)\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\, d\theta\right)^{1/p}\\ &\quad +(1-\lambda)(|\beta_2|+|\beta_4|)\left(\int_0^\omega \left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\, d\theta\right)^{1/p}\\ &\quad +(1-\lambda)(|\beta_2|+|\beta_4|)\left(\int_0^\omega \left|e^{-(\omega-\theta)\alpha_1}-e^{-(\omega-\theta)\alpha_2}\right|^p\, d\theta\right)^{1/p}.\end{aligned}$$

Using the fact that

$$\frac{1-\lambda}{-\lambda}=1+\frac{1}{-\lambda}\leqslant 1+\frac{1}{\varepsilon_0},$$

with Lemma 3.7 and Lemma 3.8, we obtain

$$\begin{aligned}\|\psi_1''-S''\|_{L^p(0,\omega)} &\leqslant \frac{2(1-\lambda)\left(|\beta_1+\beta_2|+|\beta_3+\beta_4|\right)}{\sqrt{-\lambda}^{1/p}}+\frac{8(1-\lambda)(|\beta_2|+|\beta_4|)}{\sqrt{-\lambda}^{1+1/p}}\\ &\leqslant \frac{M_3\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{-\lambda},\end{aligned}$$

where $M_3=\frac{4M_1\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\varepsilon_0}\right)}{\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1-e^{-\omega\sqrt{\varepsilon_0}}\right)}$.

Due to (39), it follows that

$$\begin{aligned}\|\psi_1''\|_{L^p(0,\omega)} &\leqslant \|\psi_1''-S''\|_{L^p(0,\omega)}+\|S''\|_{L^p(0,\omega)}\\ &\leqslant \frac{M_3+M_2}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\ &\leqslant \frac{3(M_2+M_3)}{-\lambda}\|F\|_X.\end{aligned}$$

From the Poincaré inequality, there exists $C_\omega>0$ such that

$$\|\psi_1\|_{W_0^{2,p}(0,\omega)}\leqslant C_\omega\|\psi_1''\|_{L^p(0,\omega)}\leqslant \frac{3C_\omega(M_2+M_3)}{-\lambda}\|F\|_X. \tag{40}$$

Now, we focus ourselves on $\|\psi_2\|_{L^p(0,\omega)}$. As previously, we obtain

$$\begin{aligned}\|\psi_2\|_{L^p(0,\omega)} &\leqslant |\lambda|(|\beta_1+\beta_2|+|\beta_3+\beta_4|)\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\, d\theta\right)^{1/p}\\ &\quad +|\lambda|(|\beta_1+\beta_2|+|\beta_3+\beta_4|)\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\, d\theta\right)^{1/p}\\ &\quad +|\lambda|(|\beta_2|+|\beta_4|)\left(\int_0^\omega \left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\, d\theta\right)^{1/p}\\ &\quad +|\lambda|(|\beta_4|+|\beta_2|)\left(\int_0^\omega \left|e^{-(\omega-\theta)\alpha_1}-e^{-(\omega-\theta)\alpha_2}\right|^p\, d\theta\right)^{1/p}+\|\lambda S+F_1\|_{L^p(0,\omega)}\\ &\leqslant \frac{M_2+M_3}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\ &\quad +\|\lambda S+F_1\|_{L^p(0,\omega)}.\end{aligned}$$

Moreover, from (37) and Lemma 3.6, we deduce that

$$\begin{aligned}
\|\lambda S + F_1\|_{L^p(0,\omega)} &\leqslant \frac{-\lambda\left(|J(0)|+|J(\omega)|\right)}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\,d\theta\right)^{1/p} \\
&\quad + \frac{-\lambda\left(|J(0)|+|J(\omega)|\right)}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\,d\theta\right)^{1/p} \\
&\quad + \left|1-\frac{\lambda^2}{\alpha_1^2\alpha_2^2}\right|\|F_1\|_{L^p(0,\omega)} + \frac{-\lambda}{2\sqrt{1-\lambda}}\|J\|_{L^p(0,\omega)} \\
&\leqslant \frac{-\lambda\left(|J(0)|+|J(\omega)|\right)}{\sqrt{-\lambda}^{1+1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)} + \left|1-\frac{\lambda^2}{(1-\lambda)^2}\right|\|F_1\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} \\
&\leqslant \frac{-2\lambda}{\sqrt{-\lambda}^2\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\|v\|_{L^p(0,\omega)} + \frac{1-2\lambda}{(1-\lambda)^2}\|F_1\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} \\
&\leqslant \left(\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}+1\right)\|v\|_{L^p(0,\omega)} + \left(\frac{1}{\lambda^2}+\frac{1}{-\lambda}\right)\|F_1\|_{L^p(0,\omega)} \\
&\leqslant \left(\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}+1\right)\|v\|_{L^p(0,\omega)} + \frac{\frac{1}{\varepsilon_0}+1}{-\lambda}\|F_1\|_{L^p(0,\omega)}.
\end{aligned}$$

Then, from Lemma 3.6, we obtain

$$\|\lambda S + F_1\|_{L^p(0,\omega)} \leqslant \frac{M_4}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right),$$

where $M_4 = \left(\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}+1\right)M_1 + \frac{1}{\varepsilon_0} + 1$.

Thus, it follows that

$$\begin{aligned}
\|\psi_2\|_{L^p(0,\omega)} &\leqslant \frac{M_2+M_3+M_4}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right) \\
&\leqslant \frac{3\left(M_2+M_3+M_4\right)}{-\lambda}\|F\|_X.
\end{aligned}$$

Finally, from (40), we have

$$\|(\mathcal{A}-\lambda I)^{-1}F\|_X = \|\psi_1\|_{W_0^{2,p}(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)} \leqslant \frac{M}{|\lambda|}\|F\|_X,$$

where $M = 3\left((C_\omega+1)(M_2+M_3)+M_4\right)$. □

Since $-\mathcal{A}$ is the realization of $\mathcal{L}_2$, we deduce the following corollary.

Corollary 3.11. There exist $\varepsilon_{\mathcal{L}_2} \in (0,\pi)$ small enough and $M_{\mathcal{L}_2} > 0$ such that

$$\forall\ z \in \Sigma_{\mathcal{L}_2} := \overline{B(0,\varepsilon_0)} \cup \{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| \leqslant \varepsilon_{\mathcal{L}_2}\},$$

we have

$$\left\|(\mathcal{L}_2 - zI)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{M_{\mathcal{L}_2}}{1+|z|}.$$

Therefore, assumption (H_1) in Section 2.2 is verified for $\mathcal{L}_2$ with

$$\theta_{\mathcal{L}_2} = \pi - \varepsilon_{\mathcal{L}_2}. \tag{41}$$

Remark 3.12. $\mathcal{A}$ is anti-compact; since $\sigma(-\mathcal{L}_2) = \sigma(\mathcal{A})$ then $\sigma(-\mathcal{L}_2)$ is uniquely composed by isolated eigenvalues $(\lambda_j)_{j\geqslant 1}$ such that $|\lambda_j| \to +\infty$, see Kato [10], Theorem 6.29, p. 187. More precisely, the calculus of the resolvent operator $(\mathcal{A} - \lambda I)^{-1}$ requires that, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, U_1 and U_2 defined by (21) do not vanish. Since $U_1 U_2 = 0$ is equivalent to

$$\left(\sinh(\omega\sqrt{-\lambda}) - \omega\sqrt{-\lambda}\right)\left(\sinh(\omega\sqrt{-\lambda}) + \omega\sqrt{-\lambda}\right) = 0,$$

then, using $(z_j)_{j\geqslant 1}$ defined in Section 1, we deduce that

$$\forall j \geqslant 1, \quad \lambda_j = -\frac{z_j^2}{\omega^2} \in \mathbb{C} \setminus \mathbb{R}_+.$$

Now, we prove that operator $\mathcal{A}$ has Bounded Imaginary Powers, see Definition 2.4.

Proposition 3.13. $\mathcal{A} \in \mathrm{BIP}\,(X, \theta_{\mathcal{A}})$, for any $\theta_{\mathcal{A}} \in (0, \pi)$.

Proof. We will be inspired by the method used in Labbas and Moussaoui [16] or in Labbas and Sadallah [17].

Let $\varepsilon > 0$ and $r \in \mathbb{R}$. For all $\lambda > 0$, $F_1 \in W_0^{2,p}(0,\omega)$ and $F_2 \in L^p(0,\omega)$, we have

$$\begin{aligned}\left[(\mathcal{A}+I)^{-\varepsilon+ir}\begin{pmatrix}F_1\\F_2\end{pmatrix}\right](\theta) &= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\lambda^{-\varepsilon+ir}\left[(\mathcal{A}+I+\lambda I)^{-1}\begin{pmatrix}F_1\\F_2\end{pmatrix}\right](\theta)\,d\lambda\\ &= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\lambda^{-\varepsilon+ir}\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}(\theta)\,d\lambda\end{aligned}$$

where $\Gamma_{\varepsilon,r} = \Gamma(1-\varepsilon+ir)\Gamma(\varepsilon - ir)$, see for instance Triebel [23], (6), p. 100.

Now, let us focus ourselves on the first component ψ_1 for instance. Due to Seeley [21], we only consider the convolution term in ψ_1, which is the most singular term. In our case, this term is given by

$$\begin{aligned}I_S(\theta) &= \frac{1}{4\alpha_1\alpha_2}\int_0^\omega e^{-|\theta-s|\alpha_2}\int_0^\omega e^{-|s-t|\alpha_1}G_{\lambda+1}(t)\;dtds\\ &= \frac{1}{4\alpha_1\alpha_2}\int_0^\omega e^{-|\theta-s|\alpha_2}\int_0^\omega e^{-|s-t|\alpha_1}\left(-F_2 - 2(F_1'' - F_1) + (\lambda+1)F_1\right)(t)\;dtds,\end{aligned}$$

see (26) and (27). We will use the two following extensions

$$\widetilde{G_0}(\theta) = \begin{cases} -F_2(\theta) - 2F_1''(\theta) + 2F_1(\theta), & \text{if } \theta \in [0,\omega]\\ 0, & \text{else,}\end{cases} \quad \text{with} \quad \widetilde{F_1}(\theta) = \begin{cases} F_1(\theta), & \text{if } \theta \in [0,\omega]\\ 0, & \text{else,}\end{cases}$$

and

$$E_\alpha(\theta) = e^{-|\theta|\alpha}.$$

Now, we will use the Fourier transform denoted by $\mathcal{F}_t$. We then have

$$\begin{aligned}I_{\varepsilon,r}(\theta) &:= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\lambda^{-\varepsilon+ir}I_S(\theta)\;d\lambda\\ &= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\left(E_{\alpha_2}\star\left(E_{\alpha_1}\star\left(\widetilde{G_0}+(\lambda+1)\widetilde{F_1}\right)\right)\right)(\theta)\;d\lambda\\ &= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}_t^{-1}\left(\mathcal{F}_t\left(E_{\alpha_2}\star\left(E_{\alpha_1}\star\left(\widetilde{G_0}+(\lambda+1)\widetilde{F_1}\right)\right)\right)(\xi)\right)(\theta)\;d\lambda,\end{aligned}$$

hence

$$\begin{aligned}
I_{\varepsilon,r}(\theta) &= \mathcal{F}_t^{-1}\left(\frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi)\mathcal{F}_t\left(\widetilde{G_0}+(\lambda+1)\widetilde{F_1}\right)(\xi)\ d\lambda\right)(\theta)\\
&= \mathcal{F}_t^{-1}\left(\frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi)\ d\lambda\,\mathcal{F}_t\left(\widetilde{G_0}\right)(\xi)\right)(\theta)\\
&\quad+\mathcal{F}_t^{-1}\left(\frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi)\ d\lambda\,\mathcal{F}_t\left(\widetilde{F_1}\right)(\xi)\right)(\theta).
\end{aligned}$$

We recall that

$$\mathcal{F}_t\left(E_\alpha\right)(\xi)=\frac{2\alpha}{\alpha^2+4\pi^2\xi^2},$$

here $\alpha_1=\sqrt{\lambda+1}+i$ and $\alpha_2=\sqrt{\lambda+1}-i$. Hence

$$\begin{aligned}
\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi) &= \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\frac{4\alpha_1\alpha_2}{(\alpha_1^2+4\pi^2\xi^2)(\alpha_2^2+4\pi^2\xi^2)}\\
&= \frac{\lambda^{-\varepsilon+ir}}{\alpha_1^2\alpha_2^2+4\pi^2\xi^2(\alpha_1^2+\alpha_2^2)+16\pi^4\xi^4}\\
&= \frac{\lambda^{-\varepsilon+ir}}{\lambda^2+4(1+2\pi^2\xi^2)\lambda+4(1+4\pi^4\xi^4)}\\
&= \frac{\lambda^{-\varepsilon+ir}}{(\lambda+\lambda_1)(\lambda+\lambda_2)},
\end{aligned}$$

where

$$\begin{cases}
\lambda_1=2+4\pi\xi+4\pi^2\xi^2=4\pi^2\left(\xi-\dfrac{(1+i)}{2\pi}\right)\left(\xi-\dfrac{(1-i)}{2\pi}\right)\\[2ex]
\lambda_2=2-4\pi\xi+4\pi^2\xi^2=4\pi^2\left(\xi+\dfrac{(1+i)}{2\pi}\right)\left(\xi+\dfrac{(1-i)}{2\pi}\right).
\end{cases}$$

Thus, since

$$\frac{1}{(\lambda+\lambda_1)(\lambda+\lambda_2)}=\frac{1}{\lambda_1-\lambda_2}\left(-\frac{1}{\lambda+\lambda_1}+\frac{1}{\lambda+\lambda_2}\right),$$

it follows that

$$\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi)=\frac{1}{8\pi\xi}\left(-\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1}+\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\right).$$

Then, setting

$$\sigma_1=\frac{\lambda}{\lambda_1}\quad\text{and}\quad\sigma_2=\frac{\lambda}{\lambda_2},$$

we obtain

$$\begin{aligned}
\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi)\ d\lambda &= \frac{1}{8\pi\xi}\left(-\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1}\ d\lambda+\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\ d\lambda\right)\\
&= -\frac{\lambda_1^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_1^{-\varepsilon+ir}}{\sigma_1+1}\ d\sigma_1\\
&\quad+\frac{\lambda_2^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_2^{-\varepsilon+ir}}{\sigma_2+1}\ d\sigma_2.
\end{aligned}$$

Moreover, for all $z\in\mathbb{C}\setminus\mathbb{N}^-$, where $\mathbb{N}^-$ is the set of negative integer, we have

$$\int_0^{+\infty}\frac{\sigma^{-z}}{\sigma+1}\ d\sigma=\Gamma(z)\Gamma(1-z). \tag{42}$$

It follows that

$$\int_0^{+\infty} \frac{\sigma^{-\varepsilon+ir}}{\sigma+1}\, d\sigma = \Gamma(\varepsilon - ir)\Gamma(1-\varepsilon+ir) = \Gamma_{\varepsilon,r}, \tag{43}$$

hence

$$\begin{aligned}
\frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi)\, d\lambda &= \frac{1}{8\pi\xi\,\Gamma_{\varepsilon,r}} \left(\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}\right)\Gamma_{\varepsilon,r} \\
&= \frac{1}{8\pi\xi}\left(\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}\right).
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi) &= \frac{1}{\lambda_1-\lambda_2}\left(-\frac{(\lambda+1)\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1} + \frac{(\lambda+1)\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\right) \\
&= \frac{1}{8\pi\xi}\left(-\frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_1} + \frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_2}\right) \\
&\quad + \frac{1}{8\pi\xi}\left(-\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1} + \frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\right).
\end{aligned}$$

Then, setting

$$\sigma_1 = \frac{\lambda}{\lambda_1} \quad \text{and} \quad \sigma_2 = \frac{\lambda}{\lambda_2},$$

we obtain

$$\begin{aligned}
\Upsilon &= \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2}\mathcal{F}_t\left(E_{\alpha_2}\right)(\xi)\mathcal{F}_t\left(E_{\alpha_1}\right)(\xi)\, d\lambda \\
&= -\frac{1}{8\pi\xi}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1}\, d\lambda + \frac{1}{8\pi\xi}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\, d\lambda \\
&\quad -\frac{1}{8\pi\xi}\int_0^{+\infty}\frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_1}\, d\lambda + \frac{1}{8\pi\xi}\int_0^{+\infty}\frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_2}\, d\lambda \\
&= -\frac{\lambda_1^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_1^{-\varepsilon+ir}}{\sigma_1+1}\, d\sigma_1 + \frac{\lambda_2^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_2^{-\varepsilon+ir}}{\sigma_2+1}\, d\sigma_2 \\
&\quad -\frac{\lambda_1^{1-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_1^{1-\varepsilon+ir}}{\sigma_1+1}\, d\sigma_1 + \frac{\lambda_2^{1-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_2^{1-\varepsilon+ir}}{\sigma_2+1}\, d\sigma_2.
\end{aligned}$$

Moreover, from (42) and (43), we deduce that

$$\begin{aligned}
\Upsilon &= \left(\frac{\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}}{8\pi\xi}\right)\Gamma_{\varepsilon,r} + \left(\frac{\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}}{8\pi\xi}\right)\Gamma(\varepsilon-ir-1)\Gamma(1-(\varepsilon-ir-1)) \\
&= \left(\frac{\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}}{2\pi\xi}\right)\left(\Gamma_{\varepsilon,r} + \Gamma(\varepsilon-ir-1)\Gamma(1-(\varepsilon-ir-1))\right).
\end{aligned}$$

For all $z \in \mathbb{C} \setminus \mathbb{Z}$, we have

$$\Gamma(z-1)\Gamma(1-(z-1)) = \frac{\pi}{\sin(\pi(z-1))} = -\frac{\pi}{\sin(\pi z)} = -\Gamma(z)\Gamma(1-z),$$

Setting $z = \varepsilon - ir$, with $\varepsilon \in (0,1)$, it follows that

$$\Gamma(\varepsilon-ir-1)\Gamma(1-(\varepsilon-ir-1) = -\Gamma(\varepsilon-ir)\Gamma(1-\varepsilon+ir) = -\Gamma_{\varepsilon,r},$$

hence $\Upsilon = 0$. Finally, we obtain that

$$I_{\varepsilon,r}(\theta) = \mathcal{F}_t^{-1}\left(m_\varepsilon(\xi)\mathcal{F}_t\left(\widetilde{G_0}\right)(\xi)\right)(\theta),$$

where

$$m_\varepsilon(\xi) = \frac{\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}}{8\pi\xi}.$$

Setting

$$m(\xi) := \lim_{\varepsilon\to 0} m_\varepsilon(\xi) = \frac{\lambda_2^{ir} - \lambda_1^{ir}}{8\pi\xi},$$

due to the Lebesgue's dominated convergence Theorem, it follows that

$$I_{0,r}(\theta) := \lim_{\varepsilon\to 0} I_{\varepsilon,r}(\theta) = \mathcal{F}_t^{-1}\left(m(\xi)\mathcal{F}_t\left(\widetilde{G_0}\right)(\xi)\right)(\theta).$$

Moreover, for all $x_1, x_2 \in \mathbb{R}$, we have

$$\left|e^{ix_1} - e^{ix_2}\right| \leqslant |x_1 - x_2|,$$

then, for all $\xi \in \mathbb{R}\setminus\{0\}$, we deduce that

$$|m(\xi)| = \frac{|\lambda_2^{ir} - \lambda_1^{ir}|}{8\pi|\xi|} = \frac{\left|e^{ir\ln(\lambda_2)} - e^{ir\ln(\lambda_1)}\right|}{8\pi|\xi|} \leqslant \frac{|r|\,|\ln(\lambda_2) - \ln(\lambda_1)|}{8\pi|\xi|} \leqslant \frac{|r|\left|\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)\right|}{8\pi\xi}.$$

Thus

$$\sup_{\xi\in\mathbb{R}} |m(\xi)| = \lim_{\xi\to 0} |m(\xi)| = \left|\lim_{\xi\to 0} \frac{\lambda_2^{ir} - \lambda_1^{ir}}{8\pi\xi}\right|,$$

and

$$\begin{aligned}
\lim_{\xi\to 0} \frac{\lambda_2^{ir} - \lambda_1^{ir}}{8\pi\xi} &= \lim_{\xi\to 0} 2^{ir}\left(\frac{1 + ir\left(-2\pi\xi + 2\pi^2\xi^2\right) + 2ir(ir-1)\pi^2\xi^2 + o(\xi^2)}{8\pi\xi}\right) \\
&\quad - \lim_{\xi\to 0} 2^{ir}\left(\frac{1 + ir\left(2\pi\xi + 2\pi^2\xi^2\right) + 2ir(ir-1)\pi^2\xi^2 + o(\xi^2)}{8\pi\xi}\right) \\
&= \lim_{\xi\to 0} 2^{ir}\left(\frac{-4ir\pi\xi + o(\xi^2)}{8\pi\xi}\right) \\
&= -2^{ir-1} ir.
\end{aligned}$$

Then

$$\sup_{\xi\in\mathbb{R}} |m(\xi)| = \frac{|r|}{2}.$$

We have

$$\begin{aligned}
\xi\, m'(\xi) &= \frac{2\pi\xi^2\left(ir\lambda_2^{ir-1}(-4\pi + 8\pi^2\xi) - ir\lambda_1^{ir-1}(4\pi + 8\pi^2\xi)\right) - 2\pi\xi\left(\lambda_2^{ir} - \lambda_1^{ir}\right)}{64\pi^2\xi^2} \\
&= \frac{ir}{32}\left(\lambda_2^{ir-1}(-1 + 2\pi\xi) - \lambda_1^{ir-1}(1 + 2\pi\xi)\right) - \frac{\lambda_2^{ir} - \lambda_1^{ir}}{32\pi\xi},
\end{aligned}$$

and in the same way we obtain

$$\sup_{\xi\in\mathbb{R}} |\xi\, m'(\xi)| = \lim_{\xi\to 0} |\xi\, m'(\xi)| = \left|\lim_{\xi\to 0} \xi\, m'(\xi)\right|,$$

with

$$\lim_{\xi\to 0}\xi\, m'(\xi) = \frac{ir}{32}\lim_{\xi\to 0}\left(\lambda_2^{ir-1}(-1+2\pi\xi) - \lambda_1^{ir-1}(1+2\pi\xi)\right) - \frac{\lambda_2^{ir}-\lambda_1^{ir}}{32\pi\xi},$$

where

$$\lim_{\xi\to 0}\lambda_2^{ir-1} = \lim_{\xi\to 0} 2^{ir-1}\left(1+(ir-1)\left(-2\pi\xi+2\pi^2\xi^2\right)+4ir(ir-1)\pi^2\xi^2+o(\xi^2)\right) = 2^{ir-1},$$

and

$$\lim_{\xi\to 0}\lambda_1^{ir-1} = \lim_{\xi\to 0} 2^{ir-1}\left(1+(ir-1)\left(2\pi\xi+2\pi^2\xi^2\right)+4ir(ir-1)\pi^2\xi^2+o(\xi^2)\right) = 2^{ir-1}.$$

Thus

$$\begin{aligned}\lim_{\xi\to 0}\xi\, m'(\xi) &= \frac{1}{4}\lim_{\xi\to 0}\frac{ir}{8}\left(-\left(\lambda_2^{ir-1}+\lambda_1^{ir-1}\right)+2\pi\xi\left(\lambda_2^{ir-1}-\lambda_1^{ir-1}\right)\right) - \frac{\lambda_2^{ir}-\lambda_1^{ir}}{8\pi\xi}\\ &= \frac{1}{4}\left(-\frac{2^{ir}ir}{8}+2^{ir-1}ir\right)\\ &= 3\times 2^{ir-5}ir.\end{aligned}$$

Then

$$\sup_{\xi\in\mathbb{R}}|\xi\, m'(\xi)| = \left|3\times 2^{ir-5}ir\right| = \frac{3}{32}|r|.$$

Therefore, we deduce that

$$\sup_{\xi\in\mathbb{R}}|m(\xi)| + \sup_{\xi\in\mathbb{R}}|\xi\, m'(\xi)| = \frac{|r|}{2}+\frac{3}{32}|r| = \frac{19}{32}|r|,$$

From the Mihlin Theorem, see Mihlin [18], for all $\gamma>0$, there exists $C_{\gamma,p}>0$, such that, for all $r\in\mathbb{R}$, we have

$$\|I_{0,r}(.)\|_{\mathcal{L}(X)} = \left\|\mathcal{F}_t^{-1}\left(m(\xi)\mathcal{F}_t(\widetilde{G_0})(\xi)\right)(.)\right\|_{\mathcal{L}(X)} \leqslant C_{\gamma,p}e^{\gamma|r|}.$$

Finally, for all $\gamma>0$, there exists a constant $C_{\gamma,p}>0$ such that for all $r\in\mathbb{R}$, we obtain

$$\left\|(\mathcal{A}+I)^{ir}\right\|_{\mathcal{L}(X)} \leqslant C_{\gamma,p}\, e^{\gamma|r|}.$$

Therefore, taking $\theta_{\mathcal{A}}=\gamma>0$, we have $\mathcal{A}+I\in \mathrm{BIP}\,(X,\theta_{\mathcal{A}})$ and from Theorem 2.3, p. 69 in Arendt, Bu and Haase [1], we deduce that $\mathcal{A}=\mathcal{A}+I-I\in\mathrm{BIP}\,(X,\theta_{\mathcal{A}})$. □

4 Study of the sum $\mathcal{L}_{1,\mu}+\mathcal{L}_2$

4.1 Invertibility of the closure of the sum

In this section, we will apply the results described in Section 2.2. We take $\mathcal{L}_{1,\mu}=\mathcal{M}_1$ and $\mathcal{L}_2=\mathcal{M}_2$.

Theorem 4.1. Assume that (5) holds. Then $\mathcal{L}_{1,\mu}+\mathcal{L}_2$ is closable and its closure $\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2}$ is invertible.

Proof. Assumption (H_1) is satisfied from Proposition 3.1 and Corollary 3.11, with

$$\theta_{\mathcal{M}_1}+\theta_{\mathcal{M}_2} = \varepsilon_{\mathcal{L}_{1,\mu}}+\pi-\varepsilon_{\mathcal{L}_2},$$

where it suffices to take $\varepsilon_{\mathcal{L}_2}>\varepsilon_{\mathcal{L}_{1,\mu}}$ in order to obtain $\theta_{\mathcal{M}_1}+\theta_{\mathcal{M}_2}<\pi$.

For assumption (H_2), due to Proposition 3.2, it follows that $0 \notin \sigma(\mathcal{L}_{1,\mu}) \cap \sigma(-\mathcal{L}_2)$. Moreover, from Proposition 3.1, we have

$$\sigma(\mathcal{L}_{1,\mu}) = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \pi \quad \text{and} \quad \mathrm{Re}(\sqrt{\lambda}) \leqslant \mu\},$$

and from Remark 3.12, it follows that

$$\sigma(-\mathcal{L}_2) = \{\lambda \in \mathbb{C} \setminus \mathbb{R}_+ : \sinh(\omega\sqrt{-\lambda}) = \pm\omega\sqrt{-\lambda}\} = \left\{-\frac{z_j^2}{\omega^2} \in \mathbb{C} \setminus \mathbb{R}_+ : j \in \mathbb{N} \setminus \{0\}\right\}.$$

Then, since

$$\mathrm{Re}\left(\sqrt{-\frac{z_j^2}{\omega^2}}\right) = \frac{1}{\omega}\left|\mathrm{Im}(z_j)\right|,$$

the condition $\sigma(\mathcal{L}_{1,\mu}) \cap \sigma(-\mathcal{L}_2) = \emptyset$ is fulfilled if (5) holds.

The commutativity assumption (H_3) is clearly verified since the actions of operators $\mathcal{L}_{1,\mu}$ and $\mathcal{L}_2$ are independent.

Now, applying Theorem 2.6, we obtain the result. □

Remark 4.2. We can conjecture that, for the critical case $\omega\mu = \tau$, the sum $\mathcal{L}_{1,\mu} + \mathcal{L}_2$ is not closable.

4.2 Convexity inequalities

In view to apply Corollary 2.7, we are going to verify inequality (8) in two situations.

Proposition 4.3. Let

$$\mathcal{E}_1 = W^{1,p}(0,+\infty;X) \subset \mathcal{E} = L^p(0,+\infty;X),$$

and

$$\mathcal{E}_2 = L^p\left(0,+\infty;\left[W^{3,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right] \times L^p(0,\omega)\right) \subset \mathcal{E}.$$

Then, we have

$$D\left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}\right) \subset \mathcal{E}_1 \cap \mathcal{E}_2. \tag{44}$$

Proof. Let $V \in D(\mathcal{L}_{1,\mu})$. We must prove that there exists $\delta \in (0,1)$ such that

$$\|V\|_{\mathcal{E}_1} \leqslant C\left[\|V\|_{\mathcal{E}} + \|V\|_{\mathcal{E}}^{1-\delta}\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}}^{\delta}\right].$$

For all $V \in W^{2,p}(0,+\infty;X)$, from Kato [10], inequality (1.15), p. 192, we have the convexity inequality

$$\|V'\|_{\mathcal{E}} \leqslant 2\sqrt{2}\|V\|_{\mathcal{E}}^{1/2}\|V''\|_{\mathcal{E}}^{1/2}.$$

Thus, we deduce that

$$\|V\|_{\mathcal{E}_1} = \|V\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} \leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2}\|V\|_{\mathcal{E}}^{1/2}\|V''\|_{\mathcal{E}}^{1/2}.$$

Since $\mathcal{L}_{1,\mu}$ is not invertible, we will estimate $\|V''\|_{\mathcal{E}}$ by $\|\mathcal{L}_{1,\mu}(V) - \lambda_0 V\|_{\mathcal{E}}$, where $\lambda_0 \in \rho(\mathcal{L}_{1,\mu})$. We have

$$V'' - 2\mu V' + (\mu^2 - \lambda_0)V = \mathcal{L}_{1,\mu}(V) - \lambda_0 V.$$

Then, there exists a constant $C > 0$ such that

$$\|V''\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} + \|V\|_{\mathcal{E}} \leqslant C\|\mathcal{L}_{1,\mu}(V) - \lambda_0 V\|_{\mathcal{E}},$$

hence

$$\|V''\|_{\mathcal{E}} \leqslant C\|\mathcal{L}_{1,\mu}(V) - \lambda_0 V\|_{\mathcal{E}} \leqslant C\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}} + |\lambda_0| C\|V\|_{\mathcal{E}}.$$

Thus

$$\begin{aligned}
\|V\|_{\mathcal{E}_1} = \|V\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} &\leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2}\,\|V\|_{\mathcal{E}}^{1/2}\|V''\|_{\mathcal{E}}^{1/2} \\
&\leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2C}\,\|V\|_{\mathcal{E}}^{1/2}\left(\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}} + |\lambda_0|\|V\|_{\mathcal{E}}\right)^{1/2} \\
&\leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2C}\,\|V\|_{\mathcal{E}}^{1/2}\left(\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}}^{1/2} + |\lambda_0|^{1/2}\|V\|_{\mathcal{E}}^{1/2}\right) \\
&\leqslant \left(1 + 2\sqrt{2C}\,|\lambda_0|^{1/2}\right)\|V\|_{\mathcal{E}} + 2\sqrt{2C}\,\|V\|_{\mathcal{E}}^{1/2}\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}}^{1/2}.
\end{aligned}$$

Therefore, inequality (8) is satisfied for $\delta = 1/2$ and $\mathcal{M}_1 = \mathcal{L}_{1,\mu}$. Using Corollary 2.7, we obtain

$$D\left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}\right) \subset \mathcal{E}_1.$$

Now, we must show that, for all $V \in D(\mathcal{L}_2)$, we have

$$\|V\|_{\mathcal{E}_2} \leqslant C\left[\|V\|_{\mathcal{E}} + \|V\|_{\mathcal{E}}^{1/2}\|\mathcal{L}_2(V)\|_{\mathcal{E}}^{1/2}\right].$$

To this end, it suffices to do it for $\mathcal{A}$. Set

$$\mathcal{G}_1 = \left[W^{3,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right] \times L^p(0,\omega) \subset X.$$

We must prove that

$$\forall \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}), \quad \left\|\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right\|_{\mathcal{G}_1} \leqslant C\left[\left\|\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right\|_X + \left\|\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right\|_X^{1/2}\left\|\mathcal{A}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right\|_X^{1/2}\right].$$

Here, we have

$$\begin{aligned}
\left\|\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\right\|_{\mathcal{G}_1} &= \|\psi_1\|_{W^{3,p}(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)} \\
&= \|\psi_1\|_{L^p(0,\omega)} + \|\psi_1'\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1'''\|_{L^p(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)}.
\end{aligned}$$

Set $\varphi = \psi_1''$. Then, for all $\eta > 0$, from Kato [10], inequality (1.12), p. 192, taking $n = \eta + 1$ and $b - a = \omega$, we obtain

$$\|\varphi'\|_{L^p(0,\omega)} \leqslant \frac{\omega}{\eta}\|\varphi''\|_{L^p(0,\omega)} + \frac{2}{\omega}\left(\eta + 3 + \frac{2}{\eta}\right)\|\varphi\|_{L^p(0,\omega)}.$$

It is not difficult to see that the second member is minimal when

$$\eta = \frac{\sqrt{2}}{2}\frac{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}}{\|\varphi\|_{L^p(0,\omega)}^{1/2}}.$$

Therefore, we deduce that

$$\begin{aligned}
\|\varphi'\|_{L^p(0,\omega)} &\leqslant \frac{\omega\sqrt{2}\,\|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi''\|_{L^p(0,\omega)}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} + \frac{4}{\omega}\frac{\sqrt{2}\,\|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi\|_{L^p(0,\omega)}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} \\
&\quad + \frac{\sqrt{2}}{\omega}\frac{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}\|\varphi\|_{L^p(0,\omega)}}{\|\varphi\|_{L^p(0,\omega)}^{1/2}} + \frac{6}{\omega}\|\varphi\|_{L^p(0,\omega)} \\
&\leqslant \frac{\sqrt{2}}{\omega}\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}\|\varphi\|_{L^p(0,\omega)}^{1/2} + \frac{6}{\omega}\|\varphi\|_{L^p(0,\omega)} \\
&\quad + \left(\frac{4}{\omega}\|\varphi\|_{L^p(0,\omega)} + \omega\|\varphi''\|_{L^p(0,\omega)}\right)\frac{\sqrt{2}\,\|\varphi\|_{L^p(0,\omega)}^{1/2}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} \\
&\leqslant C_\omega\left(\|\varphi\|_{L^p(0,\omega)} + \|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi''\|_{L^p(0,\omega)}^{1/2}\right).
\end{aligned}$$

Then, we have

$$\|\psi_1'''\|_{L^p(0,\omega)} \leqslant C_\omega \left(\|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1''\|^{1/2}_{L^p(0,\omega)} \|\psi_1^{(4)}\|^{1/2}_{L^p(0,\omega)} \right).$$

Hence

$$\begin{array}{rcl} \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} & \leqslant & \|\psi_1\|_{L^p(0,\omega)} + \|\psi_1'\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)} \\ & & +C_\omega \left(\|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1''\|^{1/2}_{L^p(0,\omega)} \|\psi_1^{(4)}\|^{1/2}_{L^p(0,\omega)} \right) + \|\psi_2\|_{L^p(0,\omega)} \\ & \leqslant & (1 + C_\omega) \|\psi_1\|_{W_0^{2,p}(0,\omega)} + C_\omega \|\psi_1\|^{1/2}_{W_0^{2,p}(0,\omega)} \|\psi_1^{(4)}\|^{1/2}_{L^p(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)} \end{array}$$

Now, since $\mathcal{A}$ is invertible, see Proposition 3.2, we have proved that there exists a constant C'_ω depending only on ω such that

$$\|\psi_1^{(4)}\|_{L^p(0,\omega)} \leqslant C'_\omega \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X .$$

Moreover, it follows

$$\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} \leqslant (1 + C_\omega) \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X + C_\omega C'_\omega \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|^{1/2}_X \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|^{1/2}_X .$$

Therefore, inequality (8) is satisfied for $\delta = 1/2$ and $\mathcal{M}_2 = \mathcal{L}_2$. Using Corollary 2.7, we obtain

$$D \left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right) \subset \mathcal{E}_2,$$

which gives the expected result. □

5 Back to the abstract problem

Now, we are in position to solve the following equation

$$\left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right) V + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_{2,\mu} \right) V = \mathcal{F}. \tag{45}$$

Theorem 5.1. Let $\mathcal{F} \in L^p(0, +\infty; X)$ and assume that (5) holds. Then, there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$, equation (45) has a unique strong solution $V \in L^p(0, +\infty; X)$, that is

$$\left\{ \begin{array}{l} \exists \ (V_n)_{n \geqslant 0} \in D(\mathcal{L}_{1,\mu}) \cap D(\mathcal{L}_2) : \\ V_n \underset{n \to +\infty}{\longrightarrow} V \text{ in } L^p(0, +\infty; X) \text{ and} \\ (\mathcal{L}_{1,\mu} + \mathcal{L}_2) V_n + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_{2,\mu} \right) V_n \underset{n \to +\infty}{\longrightarrow} \mathcal{F} \text{ in } L^p(0, +\infty; X), \end{array} \right. \tag{46}$$

satisfying

$$V \in W^{1,p}(0, +\infty; X) \cap L^p \left(0, +\infty; \left[W^{3,p}(0, \omega) \cap W_0^{2,p}(0, \omega) \right] \times L^p(0, \omega) \right). \tag{47}$$

Proof. Due to Theorem 4.1, if (5) holds, then $\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}$ is invertible. Thus, it follows that

$$\left[I + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_{2,\mu} \right) \left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right)^{-1} \right] \left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right) V = \mathcal{F}.$$

From (44), we deduce that $V \in D(\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2}) \subset \mathcal{E}_1 \cap \mathcal{E}_2$, that is (47) which involves that

$$(\mathcal{P}_1+\mathcal{P}_{2,\mu})(\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2})^{-1} \in \mathcal{L}(X).$$

Then, there exists $\rho_0 > 0$ small enough such that, for all $\rho \in (0, \rho_0]$, we have

$$V = \left(\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2}\right)^{-1}\left[I + k\rho^2\left(\mathcal{P}_1+\mathcal{P}_{2,\mu}\right)\left(\overline{\mathcal{L}_{1,\mu}+\mathcal{L}_2}\right)^{-1}\right]^{-1}\mathcal{F}, \tag{48}$$

which means that V is the unique strong solution of (45). □

6 Proof of Theorem 1.1

From Theorem 5.1, there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$, equation (45) has a unique strong solution $V \in L^p(0,+\infty;X)$ satisfying (47). Then, due to (46), there exists a sequence $(V_n)_{n\in\mathbb{N}} \in D(\mathcal{L}_{1,\mu}+\mathcal{L}_2)$ such that $V_n \underset{n\to+\infty}{\longrightarrow} V$ and

$$\lim_{n\to+\infty} \left(\mathcal{L}_{1,\mu}+\mathcal{L}_2\right)V_n + k\rho^2\left(\mathcal{P}_1+\mathcal{P}_{2,\mu}\right)V_n = \mathcal{F}.$$

Since $(V_n)_{n\in\mathbb{N}} \in D(\mathcal{L}_{1,\mu}+\mathcal{L}_2)$, then the previous equality can be written as

$$\begin{cases} \lim\limits_{n\to+\infty} \left(V_n''(t) - \mathcal{A}V_n(t) - \mathcal{F}_n(t)\right) = 0 \\ \lim\limits_{n\to+\infty} V_n(0) = 0, \quad \lim\limits_{n\to+\infty} V_n(+\infty) = 0, \end{cases} \tag{49}$$

where

$$\mathcal{F}_n(t) = k\rho^2 e^{-2t}\mathcal{A}_0 V_n(t) + k\rho^2 e^{-2t}\left[(\mathcal{B}_{2,\mu}V_n)\right](t) + 2\mu V_n'(t) - \mu^2 V_n(t) + \mathcal{F}(t).$$

Since $V_n \underset{n\to+\infty}{\longrightarrow} V$ in $\mathcal{E}$ and V satisfies (47), we deduce that

$$\lim_{n\to+\infty} V_n(0) = V(0) = 0 \quad \text{with} \quad \lim_{n\to+\infty} V_n(+\infty) = V(+\infty) = 0,$$

and

$$\lim_{n\to+\infty} \mathcal{F}_n(t) = \mathcal{F}_\infty(t) \in L^p(0,+\infty;X),$$

where

$$\mathcal{F}_\infty(t) = k\rho^2 e^{-2t}\mathcal{A}_0 V(t) + k\rho^2 e^{-2t}\left[(\mathcal{B}_{2,\mu}V)\right](t) + 2\mu V'(t) - \mu^2 V(t) + \mathcal{F}(t).$$

Thus, problem (49) can be written as follows

$$\begin{cases} \lim\limits_{n\to+\infty} \left(V_n''(t) - \mathcal{A}V_n(t)\right) = \mathcal{F}_\infty(t) \\ V(0) = 0, \quad V(+\infty) = 0. \end{cases}$$

Moreover, from Proposition 3.13, $\mathcal{A} \in \mathrm{BIP}\,(X, \theta_\mathcal{A})$, with $\theta_\mathcal{A} \in (0,\pi)$ and due to Haase [9], Proposition 3.2.1, e), p. 71, it follows that $\sqrt{\mathcal{A}} \in \mathrm{BIP}\,(X, \theta_\mathcal{A}/2)$ with $\theta_\mathcal{A}/2 \in (0, \pi/2)$. Therefore, due to Eltaief and Maingot [6], Theorem 2, p. 712, with $L_1 = L_2 = -\sqrt{\mathcal{A}}$, there exists a unique classical solution to the following problem

$$\begin{cases} \mathcal{V}''(t) - \mathcal{A}\mathcal{V}(t) = \mathcal{F}_\infty(t) \\ \mathcal{V}(0) = 0, \quad \mathcal{V}(+\infty) = 0, \end{cases}$$

that is

$$\mathcal{V} \in W^{2,p}(0,+\infty;X) \cap L^p(0,+\infty;D(\mathcal{A})).$$

Thus, it follows that

$$\lim_{n\to+\infty}\left(V_n''(t)-\mathcal{A}V_n(t)\right)=\mathcal{V}''(t)-\mathcal{A}\mathcal{V},$$

hence

$$\lim_{n\to+\infty}\Big[(V_n(t)-\mathcal{V}(t))''-\mathcal{A}\left(V_n(t)-\mathcal{V}(t)\right)\Big]=0.$$

Now, set

$$\left\{\begin{array}{lll} D(\delta_2) & = & \{\varphi\in W^{2,p}(0,+\infty;X):\varphi(0)=\varphi(+\infty)=0\} \\ \delta_2\varphi & = & \varphi'', \quad \varphi\in D(\delta_2). \end{array}\right.$$

Then

$$0=\lim_{n\to+\infty}\Big[(V_n(t)-\mathcal{V}(t))''-\mathcal{A}\left(V_n(t)-\mathcal{V}(t)\right)\Big]=\lim_{n\to+\infty}-\left(-\delta_2+\mathcal{A}\right)\left(V_n(t)-\mathcal{V}(t)\right). \tag{50}$$

From Prüss and Sohr [20], Theorem C, p. 166-167, it follows that $-\delta_2\in\text{BIP}\,(X,\theta_{\delta_2})$, for every $\theta_{\delta_2}\in(0,\pi)$ and due to Proposition 3.13, $\mathcal{A}\in\text{BIP}\,(X,\theta_{\mathcal{A}})$, for all $\theta_{\mathcal{A}}\in(0,\pi)$. Thus, since $-\delta_2$ and $\mathcal{A}$ are resolvent commuting with $\theta_{\delta_2}+\theta_{\mathcal{A}}<\pi$, from Prüss and Sohr [19], Theorem 5, p. 443, we obtain that

$$-\delta_2+\mathcal{A}\in\text{BIP}\,(X,\theta),\quad \theta=\max(\theta_{\delta_2},\theta_{\mathcal{A}}).$$

Moreover, due to Proposition 3.2, we have $0\in\rho(\mathcal{A})$, then we deduce from Prüss and Sohr [19], remark at the end of p. 445, that $0\in\rho(\delta_2+\mathcal{A})$. Therefore, due to (50), we obtain that

$$\lim_{n\to+\infty}V_n(t)-\mathcal{V}(t)=0,$$

hence, since $V_n\underset{n\to+\infty}{\longrightarrow}V$, by uniqueness of the limit, we deduce that

$$V=\mathcal{V}\in W^{2,p}(0,+\infty;X)\cap L^p(0,+\infty;D(\mathcal{A})).$$

This prove that $\mathcal{L}_{1,\mu}+\mathcal{L}_2$ is closed and that $V\in D(\mathcal{L}_{1,\mu}+\mathcal{L}_2)$.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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