On a generalized diffusion problem: a complex network approach

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Abstract

In this paper, we propose a new approach for studying a generalized diffusion problem, using complex networks of reaction-diffusion equations. We model the biharmonic operator by a network, based on a finite graph, in which the couplings between nodes are linear. To this end, we study the generalized diffusion problem, establishing results of existence, uniqueness and maximal regularity of the solution *via* operator sums theory and analytic semigroups techniques. We then solve the complex network problem and present sufficient conditions for the solutions of both problems to converge to each other. Finally, we analyze their asymptotic behavior by establishing the existence of a family of exponential attractors.
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1 Introduction

In this article, we propose a new approach to study parabolic problems. This new methodology aims to establish a relationship between a linear high order parabolic problem, set on a bounded open set and a complex network problem of reduced order with linear couplings, supported by a finite graph.

Establishing a correspondence between a high order parabolic problem and a complex network of reduced order systems can be of great interest for many applications. For instance, the qualitative analysis of neural systems described by complex networks of reaction-diffusion equations could benefit from the knowledge of the biharmonic operator. In return, the biharmonic equation could be studied through the complex network framework and emergent properties are likely to be exhibited. Hence we believe that our method can lead to significant progress in the analysis of parabolic problem of various types.

Here, we consider the following generalized diffusion initial-value problem

$$\left\{\begin{array}{rcll} \dfrac{\partial u}{\partial t}(x,\,t) & = & -\Delta^2 u(x,\,t) + r\Delta u(x,\,t) + f(x,\,t) & (x,t) \in \Omega \times \mathbb{R}_+^*, \\ u(x,\,t) & = & \Delta u(x,\,t) = 0 & (x,\,t) \in \partial\Omega \times \mathbb{R}_+^*, \\ u(x,\,0) & = & u_0(x) & x \in \Omega. \end{array}\right. \tag{1}$$

As a model of Ω, we will consider a rectangular domain in the form $\Omega = \prod_{i=1}^{d}(a_i, b_i)$, d being a positive integer, $(a_i, b_i) \subset \mathbb{R}$; $f \in L^p(\Omega, \mathbb{R}_+^*)$, $p \in (1, +\infty)$, models a source term and r is a positive coefficient. In this problem, u models a density of particles (e.g. human beings, animals, dust, *etc.*) living in Ω, subject to external forces modeled by f and to Dirichlet boundary condition. The second order term Δu models the short range diffusion, whereas the biharmonic term $\Delta^2 u$ models the long range diffusion, which can be interpreted as the diffusion of u in the "neighborhood of the neighborhood" of each point of Ω. Indeed, the Laplace operator, obtained by the simple Fick diffusion modeling fails to reproduce at a refined level spatial effects as, for example, complex cell motion phenomena [8, 11, 31, 37]. However, the Landau-Ginzburg free energy functional approach leads to the superposition of the Laplace operator and the biharmonic operator as in problem (1). In a forthcoming work we aim to focus on non linear parabolic problems with applications in neuroscience.

Many articles have been devoted to studying biharmonic equations by splitting methods, leading to the study of coupled equations of second order [18, 44]. Finite differences schemes have also been studied for numerical computing of the solutions of equations involving biharmonic terms [5, 9, 26, 20]. Many theoretical results of existence, uniqueness and regularity have been obtained for fourth order problems like Cahn-Hilliard equation, see for instance [10, 17, 19, 33, 36]. More recently, problem (1) has been studied in a general Banach space setting for the linear stationary case [28, 29, 46]. Here, we aim to improve and extend the results of the latter papers to the non-stationary case. We present novel results for the parabolic problem (1), with existence, uniqueness and maximal regularity of the solutions in L^p-spaces.

Additionally, we propose to approximate problem (1) by a complex network of reaction-diffusion equations set in L^p-spaces, $p \in (1, +\infty)$. Let us describe the statement of the network problem. First of all, we split the domain Ω into a finite set of non-empty open sub-domains such that

$$\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\omega_i}, \quad \omega_i \cap \omega_j = \varnothing \text{ if } i \neq j, \quad \omega_i \neq \varnothing, \quad 1 \leq i \leq n, \tag{2}$$

as depicted in figure 1.

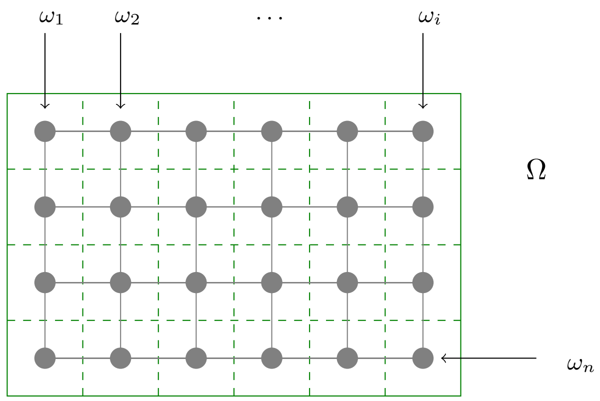

Figure 1: Splitting of a rectangular domain Ω into a grid of sub-domains $\omega_1, \ldots, \omega_n$ in the dimension case $N = 2$.

We assume that there exists a family of homeomorphisms h_i, $1 \leq i \leq n$, between ω_i and a generic open domain $\omega \subset \mathbb{R}^d$, such that two distinct sub-domains ω_i and ω_j $(i \neq j)$ may share a part of their boundaries. In this way, we have

$$\overline{\Omega} = \bigcup_{i=1}^{n} \overline{h_i^{-1}(\omega)}. \tag{3}$$

Since Ω admits a rectangular shape, then the homeomorphisms h_i, $1 \le i \le n$, are defined by elementary translations. We emphasize that the problem should not be seen as a domain decomposition problem (see for instance [14, 47] and references therein cited), since we do not impose interface conditions. For each $i \in \{1, \dots, n\}$, we denote by $\mathcal{N}_i$ the subset of indices corresponding to the neighbors of ω_i. The choice of $\mathcal{N}_i$ is not necessarily determined by the parts of the boundary $\partial\omega_i$ shared with other sub-domains ω_j, $i \neq j$; we shall see in the final section that the relevant choice of the neighbors can be found thanks to a finite differences approach. Finally, we assume that the boundary of ω_i can be split into two parts

$$\partial\omega_i = \Gamma_i^D \cup \Gamma_i^N, \quad \Gamma_i^D \cap \Gamma_i^N = \varnothing, \quad 1 \le i \le n, \tag{4}$$

which we shall associate to splitting boundary conditions, with Dirichlet boundary condition on Γ_i^D and Neumann boundary condition on Γ_i^N, $1 \le i \le n$. Hence, the complex network of reaction-diffusion equations can be written as follows

$$\begin{cases} \dfrac{\partial v_i}{\partial t}(\xi, t) = r\Delta v_i(\xi, t) + \delta_i\Big(v_i(\xi, t), \{v_j(\xi, t)\}_{j\in\mathcal{N}_i}\Big) + f_i(\xi, t) & (\xi, t) \in \omega \times \mathbb{R}_+^*, \\ \dfrac{\partial v_i}{\partial \nu}(\xi, t) = 0 & (\xi, t) \in \gamma_i^N \times \mathbb{R}_+^*, \\ v_i(\xi, t) = 0 & (\xi, t) \in \gamma_i^D \times \mathbb{R}_+^*, \\ v_i(\xi, 0) = v_{i,0}(\xi) & \xi \in \omega, \end{cases} \tag{5}$$

for all $i \in \{1, \dots, n\}$, where $f_i \in (L^p(\omega))^n$ with $p \in (1, +\infty)$, models the source term in ω, ν denotes the outer normal of $\partial\omega$, $v_{i,0}$ are smooth initial conditions and δ_i corresponds to a linear coupling operator, $1 \le i \le n$, whose we will precise the form in section 4, where we handle this problem in $Y = \left(L^2(\omega)\right)^n$ using semigroups methods.

Reaction-diffusion equations or systems have produced a huge literature, partly due to the richness of the dynamics of their solutions, which can for example exhibit traveling waves, Hopf or Turing bifurcations [25, 32, 38, 41, 43]. Complex networks of such equations have been recently analyzed, for instance in [1, 7], where the authors investigate the possibility to extend to infinite dimension the classical problematics of complex networks. Among them, one can quote the synchronization topic or the relationship between the topology of the network, the internal dynamics of its nodes and the global dynamics [2, 3, 6, 21, 40]. However, the approximation of a generalized diffusion problem by a complex network of reaction-diffusion equations represents a novelty at our knowledge.

This paper is organized as follows. In the next section, we recall some basics concerning Sobolev spaces, interpolation spaces and bounded imaginary power operators. In section 3, we present new results on the study of the generalized diffusion equation (1), with existence, uniqueness and maximal regularity theorems in a Banach space setting. We then prove the existence and uniqueness of a local in time solution to the complex network problem (5) in section 4; we show how to recover a function defined on the initial domain Ω and we establish energy estimates which imply the global existence of the solutions and the existence of exponential attractors. Finally, we investigate sufficient conditions for the convergence of the approximation problem, from which we deduce the existence of exponential attractors for the semi-flow induced by the initial generalized diffusion problem.

2 Preliminaries

In this section, we present the basic notations and material which we shall use in our paper.

2.1 Functional spaces, interpolation spaces

Throughout this paper, we will use the classical notations for Lebesgue spaces $L^p(\Omega)$, $L^p(a,b,X)$ and Sobolev spaces $W^{k,p}(\Omega)$, where Ω denotes an open bounded domain in $\mathbb{R}^d$, $p \in [1,+\infty]$, $a,b \in \mathbb{R} \cup \{\pm\infty\}$, X is a Banach space and $k \in \mathbb{N}$. Those functional spaces are Banach spaces whose norms will be denoted $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{L^p(a,b,X)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ respectively. For $p=2$, we simply note $H^k(\Omega) = W^{k,2}(\Omega)$; $H^k(\Omega)$ is a Hilbert space whose inner product will be denoted $(\cdot\,,\,\cdot)_{H^k(\Omega)}$.

Our reasonings below will use the theory of interpolation spaces, hence we recall the following definition.

Definition 2.1. Let X be a Banach space, and $T : \mathcal{D}(T) \subset X \longrightarrow X$ a linear operator such that

$$(0,+\infty) \subset \rho(T) \quad \text{and} \quad \exists\, C > 0 : \forall\, t > 0, \quad \|t(T - tI)^{-1}\|_{\mathcal{L}(X)} \leqslant C. \tag{6}$$

Then, from [23], p. 665, Teorema 3, for $\theta \in (0,1)$ and $q \in [1,+\infty]$, we can define the real interpolation space

$$(\mathcal{D}(T), X)_{\theta,q} = \left\{\psi \in X : t \longmapsto t^{1-\theta}\|T(T - tI)^{-1}\psi\|_X \in L^q_*(0,+\infty)\right\},$$

where $L^q_*(0,+\infty)$ is defined for instance in [23], p.663. In [48], this space is denoted by $(X, \mathcal{D}(T))_{1-\theta,q}$.

Note that the general situation of the real interpolation space $(X_0, X_1)_{\theta,q}$ with X_0, X_1 two Banach spaces such that $X_0 \hookrightarrow X_1$, is described for instance in [35] or [48].

2.2 Standard inequalities

For convenience, we recall a Gronwall type lemma and the Poincaré inequality [49].

Lemma 2.2 (Gronwall lemma)**.** *Assume that*

$$\phi'(t) + a\phi(t) \le b, \quad 0 < t \le T,$$

where ϕ is a continuous function on $[0,\, T]$, continuously differentiable on $(0,\, T]$, $a > 0$ and $b > 0$. Then

$$\phi(t) \le e^{-at}\phi(0) + \frac{b}{a}, \quad 0 < t \le T.$$

Let γ denote a trace operator defined from $H^1(\Omega)$ to $L^2(\partial\Omega)$.

Theorem 2.3 (Poincaré inequality)**.** *Let Ω denote an open bounded domain with regular boundary $\partial\Omega$, and $\Gamma^D \subset \partial\Omega$, $\Gamma^D \neq \varnothing$. There exists a positive constant C such that*

$$\|u\|_{L^2(\Omega)} \le C\,\|\nabla u\|_{L^2(\Omega)}$$

for all $u \in \mathring{H}^1_D(\Omega)$, where

$$\mathring{H}^1_D(\Omega) = \{\psi \in H^1(\omega)\ ;\ \gamma\psi = 0 \ \text{on}\ \Gamma^D\}.$$

2.3 The class of Bounded Imaginary Power operators

We continue with some definitions on sectorial operators, BIP operators and UMD spaces.

Definition 2.4. A closed linear operator T_1 defined in a Banach space X is said to be sectorial of angle $\alpha \in [0, \pi)$, and we write $T_1 \in \mathrm{Sect}(\alpha)$, if

$$(i) \quad \sigma(T_1) \subset \overline{S_\alpha},$$

$$(ii) \quad \forall\, \alpha' \in (\alpha, \pi), \quad \sup\left\{ \|\lambda(\lambda I - T_1)^{-1}\|_{\mathcal{L}(X)} : \ \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}} \right\} < \infty,$$

where

$$S_\alpha = \begin{cases} \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \alpha\}, & \text{if } \alpha > 0, \\ (0, +\infty), & \text{if } \alpha = 0. \end{cases} \tag{7}$$

The previous definition can be found in [24], p. 19.

It is known that any injective sectorial operator T_1 admits imaginary powers T_1^{is}, $s \in \mathbb{R}$, but, in general, T_1^{is} is not bounded, see for instance [27], p. 342.

Definition 2.5. Let X be a Banach space and $\theta \in [0, \pi)$. We denote by $\mathrm{BIP}(X, \theta)$, the class of sectorial injective operators T_1 such that

$$(i) \quad \overline{\mathcal{D}(T_1)} = \overline{R(T_1)} = X,$$

$$(ii) \quad \forall\, s \in \mathbb{R}, \quad T_1^{is} \in \mathcal{L}(X),$$

$$(iii) \quad \exists\, C \geq 1,\ \forall\, s \in \mathbb{R}, \quad ||T_1^{is}||_{\mathcal{L}(X)} \leq C e^{|s|\theta}.$$

This definition can be found in [39], p. 430. In this case, it holds that $\overline{\mathcal{D}(T_1) \cap R(T_1)} = X$, see for instance [24], proof of Proposition 3.2.1.

In this paper, we will use the well-known Dore-Venni theorem, see [16] and its generalization in [39], which needs to consider a UMD space X.

Definition 2.6. A Banach space X is a UMD space if and only if for some $p \in (1, +\infty)$ and thus for all p, the Hilbert transform is bounded from $L^p(\mathbb{R}, X)$ into itself.

Note that, in our case, we will consider $X = L^p(\Omega)$ with $p \in (1, +\infty)$ thus, from [42], Proposition 3, it holds that X is a UMD space.

2.4 Trace and regularity lemmas

In this section, we recall some known trace and regularity results.

Lemma 2.7 ([29])**.** *Let T_2 be a linear operator satisfying (6). Let u be such that*

$$u \in W^{n,p}(a, b; X) \cap L^p(a, b; \mathcal{D}(T_2)),$$

where $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N} \setminus \{0\}$ and $p \in (1, +\infty)$. Then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $s \in \{a, b\}$, we have

$$u^{(j)}(s) \in (\mathcal{D}(T_2), X)_{\frac{j}{n} + \frac{1}{np}, p}.$$

This result is proved in [22], Teorema 2'.

Lemma 2.8 ([12, 48])**.** *Let $\psi \in X$ and T_3 be a generator of a bounded analytic semigroup in X. Then, the two next properties are equivalent:*

1. $x \mapsto T_3 e^{(x-a)T_3}\psi \in L^p(a,b;X)$,

2. $\psi \in (\mathcal{D}(T_3), X)_{\frac{1}{p},p}$.

This result is proved in [48], Theorem, p. 96.

Lemma 2.9 ([16]). *Let $T_4 \in BIP(X,\theta)$ with $\theta \in (0,\pi/2)$, and $g \in L^p(a,b;X)$. Then, for almost every $x \in (a,b)$, we have*

$$\int_a^x e^{-(x-s)T_4} g(s)\, ds \in \mathcal{D}(T_4) \qquad \text{and} \qquad \int_x^b e^{-(s-x)T_4} g(s)\, ds \in \mathcal{D}(T_4).$$

Moreover,

$$\begin{aligned} x &\mapsto T_4 \int_a^x e^{-(x-s)T_4} g(s)\, ds \in L^p(a,b;X) \\ x &\mapsto T_4 \int_x^b e^{-(s-x)T_4} g(s)\, ds \in L^p(a,b;X). \end{aligned}$$

This result is proved in [16], Theorem 3.2.

3 Analysis of the generalized diffusion equation

3.1 Operational formulation

We set $X = L^p(\Omega)$, $p \in (1,+\infty)$. Let $r > 0$ and $f \in L^p(\mathbb{R}_+ \times \Omega)$, with $p \in (1,+\infty)$. Our aim in this section is to study the following parabolic problem

$$(P_{pde}) \begin{cases} \dfrac{\partial u}{\partial t} + \Delta^2 u - r\Delta u = f & \text{in } \Omega \times (0,+\infty) \\ u(x,0) = u_0(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in \mathbb{R}_+ \\ \Delta u(x,t) = 0, & x \in \partial\Omega,\ t \in \mathbb{R}_+, \end{cases}$$

where $u_0 \in L^p(\Omega)$, with $p \in (1,+\infty)$.

We handle the latter problem in $(0,T] \times \Omega$, with $T > 0$ instead of $(0,+\infty) \times \Omega$. To this end, let us define the following linear operator in $\mathbb{R}^d$:

$$\begin{cases} \mathcal{D}(\mathcal{A}) = \{\psi \in W^{4,p}(\Omega) : \psi = \Delta\psi = 0 \text{ on } \partial\Omega\} \\ \forall \psi \in \mathcal{D}(\mathcal{A}), \quad \mathcal{A}\psi = \Delta^2\psi - r\Delta\psi. \end{cases} \tag{8}$$

Thus, using operator $\mathcal{A}$, problem (P_{pde}) can be rewritten

$$(P) \begin{cases} u'(t) + \mathcal{A}u(t) = f(t), \quad t \in (0,T] \\ u(0) = u_0 \end{cases}$$

where $f \in L^p(0,T;L^p(\Omega))$ and $p \in (1,+\infty)$, with $u(t) = u(\cdot,t)$ and $f(t) = f(\cdot,t)$. We will search a classical solution of problem (P), that is, a solution u such that

$$u \in W^{1,p}(0,T;X) \cap L^p(0,T;\mathcal{D}(\mathcal{A})).$$

3.2 Existence, uniqueness and optimal regularity

Let us introduce the linear operators A_i, for all $i \in \{1, ..., k\}$, by setting

$$\begin{cases} \mathcal{D}(A_i) = \{\varphi \in W^{2,p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\} \\ A_i\varphi = -\dfrac{\partial^2 \varphi}{\partial x_i^2}, \quad \varphi \in \mathcal{D}(A_i). \end{cases} \tag{9}$$

Moreover, we also define the linear operator A by

$$\begin{cases} \mathcal{D}(A) = \{\varphi \in W^{2,p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\} \\ A\varphi = -\Delta\varphi, \quad \varphi \in \mathcal{D}(A). \end{cases} \tag{10}$$

Proposition 3.1. *For all $i \in \{1, ..., k\}$ and all $k \in \mathbb{N} \setminus \{0\}$, the linear operators A_i, defined by (9), and A defined by (10), satisfy the following properties:*

1. *$A_i \in BIP(X, \theta_i)$, for all $\theta_i \in (0, \pi)$,*
2. *$A^k \in BIP(X, \theta)$, for all $k \in \mathbb{N}$ and all $\theta \in (0, \pi)$.*

Proof.

1. It is well-known that, for all $i \in \{1, ..., k\}$, $A_i \in \text{Sect}(0)$ and $0 \in \rho(A_i)$. Moreover, since X is reflexive and from [30], Proposition 3.1, p. 191, for all $\theta_i > 0$, we obtain

$$A_i \in \text{BIP}(X, \theta_i).$$

   In particular, without lost of generality, we could take each θ_i as small as we want.

2. When $k = 0$, we have $A^k = A^0 = I$, where I denotes the identity operator. It is known that $I \in \text{Sect}(0)$ with $\sigma(I) = \{1\}$. Thus, since $I \in \mathcal{L}(X)$, we can deduce that $I \in \text{BIP}(X, \theta)$, for all $\theta > 0$.

   Let $j \in \{1, ..., k\}$ such that $j \neq i$. Since the actions of A_i and A_j are independents and since A_i and A_j have the same domain, then we have

$$A_i A_j = A_j A_i. \tag{11}$$

   Then, from (11) and Theorem 5 in [39], we obtain that

$$A = \sum_{i=1}^{k} A_i \in \text{BIP}(X, \theta_A), \tag{12}$$

   where $\theta_A = \max_{i \in \{1,...,k\}} \theta_i$. Thus, since each θ_i can be as small as we want, then $\theta_A > 0$ can be as small as we want.

   Let $k \in \mathbb{N} \setminus \{0\}$, since $0 \in \rho(A)$, from (12) and [39], Corollary 3, p. 444, one has

$$A^k \in \text{BIP}(X, k\theta_A).$$

   Moreover, since θ_A can be as small as we want, we deduce that $k\theta_A > 0$ can be as small as we want. This completes the proof.

□

Proposition 3.2. *The linear operator $\mathcal{A}$ defined by (8) satisfies the two following properties:*

1. $0 \in \rho(\mathcal{A})$,

2. $\mathcal{A} \in BIP(X, \theta)$, *for all* $\theta \in (0, \pi)$.

In particular, since $\theta = 2\theta_A$, *we have* $\mathcal{A} \in BIP(X, 2\theta_A)$, *for all* $\theta_A \in (0, \pi)$.

Proof.

1. The result follows from [29], Theorem 2.2, p. 355 and Remark 2.6, p. 357.

2. Let $r > 0$. From Proposition 3.1 and [39], Corollary 1, p. 435, for all $\theta_A > 0$, we have

$$rA \in \text{BIP}(X, \theta_A).$$

From Proposition 3.1, for all $\theta > 0$, we deduce

$$A^2 \in \text{BIP}(X, \theta).$$

Note that, from [39], Corollary 3, p. 444, we obtain $\theta = 2\theta_A$. Finally, from [39], Theorem 5, p. 443, we obtain

$$\mathcal{A} = A^2 + rA \in \text{BIP}(X, \theta).$$

□

Theorem 3.3. *Let* $f \in L^p(0, T; X)$, $p \in (1, +\infty)$. *There exists a unique classical solution given by*

$$u(t) = e^{-t\mathcal{A}} u_0 + \int_0^t e^{-(t-s)\mathcal{A}} f(s)\ ds, \tag{13}$$

to problem:

$$(P) \begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in (0, T] \\ u(0) = u_0, \end{cases}$$

if and only if

$$u_0 \in (\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p}, p}. \tag{14}$$

Proof. From Proposition 3.2, we deduce that for all $\theta \in (0, \pi)$, $\mathcal{A} \in \text{BIP}(X, \theta)$. In particular we could take $\theta < \pi/2$. Thus, $-\mathcal{A}$ generate an analytic semigroup $\left(e^{-t\mathcal{A}}\right)_{t \geqslant 0}$. Then, since $f \in L^p(0, T; X)$, $p \in (1, +\infty)$, from [13], Lemma 2.1, p. 208, there exists a unique solution of problem (P) given by (13):

$$u(t) = e^{-t\mathcal{A}} u_0 + \int_0^t e^{-(t-s)\mathcal{A}} f(s)\ ds.$$

If u, given by (13), is a classical solution of problem (P), then

$$u \in W^{1,p}(0, T; X) \cap L^p\left(0, T; \mathcal{D}(\mathcal{A})\right).$$

It follows from Lemma 2.7 that

$$u(0) = u_0 \in (\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p}, p}.$$

If $u_0 \in (\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p}, p}$, since the unique solution u is given by (13), it remains to show that u is a classical solution. From Lemma 2.7, we deduce that

$$t \longmapsto e^{-t\mathcal{A}} u_0 \in W^{1,p}(0, T; X) \cap L^p\left(0, T; \mathcal{D}(\mathcal{A})\right).$$

Finally, from Lemma 2.9, we obtain that

$$t \longmapsto \int_0^t e^{-(t-s)\mathcal{A}} f(s)\, ds \in W^{1,p}(0,T;X) \cap L^p\left(0,T;\mathcal{D}(\mathcal{A})\right).$$

□

From [15], Theorem 2.4, p. 28, since $-A$ has negative exponential type, we deduce the following result on $\mathbb{R}_+$.

Corollary 3.4. *Let $f \in L^p(\mathbb{R}_+;X)$, $p \in (1,+\infty)$. There exists a unique classical solution, given by (13), to problem*

$$(P) \begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in (0,+\infty) \\ u(0) = u_0, \end{cases}$$

if and only if (14) holds.

As a consequence of Theorem 3.3 and Corollary 3.4, we deduce the following result for problem (P_{pde}).

Corollary 3.5. *Let $r \in \mathbb{R}_+ \setminus \{0\}$ and $f \in L^p(\mathbb{R}_+ \times \Omega)$ where $p \in (1,+\infty)$ such that $p > \frac{d}{4}+1$. Then, there exists a unique solution u of (P_{pde}), such that*

$$u \in W^{1,p}(\mathbb{R}_+, L^p(\Omega)) \cap L^p(\mathbb{R}_+, W^{4,p}(\Omega)),$$

if and only if

$$u_0 \in \{\varphi \in W^{4-\frac{4}{p},p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\}.$$

Remark 3.6. Taking into account the results of Theorem 3.3 and Corollary 3.4, we can also obtain anisotropic result by considering $f \in L^p(0,T;L^q(\Omega))$ with $p,q \in (1,+\infty)$.

Proof. Let $(t,x) \in (0,+\infty) \times \Omega$. Set $X = L^p(\Omega)$. Using the linear operator $\mathcal{A}$ defined by (8), we obtain that problem (P_{pde}) becomes problem (P) when $t \in \mathbb{R}_+$. From Corollary 3.4, there exists a unique classical solution of problem (P) if and only if (14) holds.

Now, it remains to show that if u_0 satisfies (14), then the classical solution u satisfies

$$u \in W^{1,p}(\mathbb{R}_+, L^p(\Omega)) \cap L^p(\mathbb{R}_+, W^{4,p}(\Omega)).$$

To this end, we explicit the interpolation space that appears in (14). We have

$$(\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p},p} = \Big(\{\varphi \in W^{4,p}(\Omega) : \varphi = \Delta\varphi = 0 \text{ on } \partial\Omega\}, L^p(\Omega)\Big)_{\frac{1}{p},p};$$

moreover, from [22], proposizione 3, p. 683 and [48], Theorem 1, p. 317, since $4 - \frac{4}{p}$ is never integer, we have

$$\Big(W^{4,p}(\Omega), L^p(\Omega)\Big)_{\frac{1}{p},p} = B^{4(1-\frac{1}{p})}_{p,p}(\Omega) = B^{4-\frac{4}{p}}_{p,p}(\Omega) = W^{4-\frac{4}{p},p}(\Omega). \tag{15}$$

Set $\nu = 4 - \dfrac{4}{p} - \dfrac{d}{p} = 4 - \dfrac{d+4}{p}$. Since $p > \dfrac{d}{4}+1$, we have $\nu > 0$. From the Sobolev embedding theorem, see [48], section 4.6.1, (e), p. 327-328, we have:

$$W^{4-\frac{4}{p},p}(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

Thus, we deduce that

$$(\mathcal{D}(\mathcal{A}), X)_{\frac{1}{p},p} = \left\{\varphi \in W^{4-\frac{4}{p},p}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\right\}. \tag{16}$$

□

4 Analysis of the complex network problem

In this section, our aim is to study the complex network problem of reaction-diffusion equations approximating the generalized diffusion equation (1).

4.1 Setting of the complex network problem

As mentioned in our introduction, we suppose that the domain Ω can be split into a finite set of non-empty open sub-domains

$$\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\omega_i}, \quad \omega_i \cap \omega_j = \varnothing \ \text{ if } \ i \neq j, \quad \omega_i \neq \varnothing, \quad 1 \leq i \leq n. \tag{17}$$

For each $i \in \{1, \ldots, n\}$, we denote by $\mathcal{N}_i$ a subset of indices corresponding to the neighbors of ω_i. We recall that the choice of $\mathcal{N}_i$ is not necessarily determined by the parts of the boundary $\partial\omega_i$ shared with other sub-domains ω_j, $i \neq j$. The sub-domains ω_i and the sets $\mathcal{N}_i$, $1 \leq i \leq n$, generate a graph $\mathcal{G}$, that underlies the complex network of reaction-diffusion systems that we construct. Furthermore, we assume that there exist an open bounded domain $\omega \subset \mathbb{R}^d$ and a family of homeomorphisms h_i, $1 \leq i \leq n$, defined on ω_i with values in ω. We denote by h_i^{-1} the inverse homeomorphism of h_i. The relationship between Ω and ω has been given previously (see equation (3)). Every function ϕ defined on $\Omega \times J$, $J \subset \mathbb{R}$, with values in $\mathbb{R}$ determines a function $(\phi_1, \ldots, \phi_n)$ defined on $\omega \times J$ with values in $\mathbb{R}^n$, such that

$$\phi_i(\xi, t) = \phi(h_i^{-1}(\xi), t), \quad (\xi, t) \in \omega \times J, \quad 1 \leq i \leq n.$$

In particular, the source function f defined on $\Omega \times (0, T]$ involved in problem (1), determines a function $(f_1, \ldots, f_n)$, as well as the initial condition u_0 of problem (1) induces a function $(u_{0,1}, \ldots, u_{0,n})$. Conversely, every continuous function $(\phi_1, \ldots, \phi_n)$ defined on $\omega \times J$, $J \subset \mathbb{R}$, with values in $\mathbb{R}^n$ determines a function ϕ defined on $\Omega \times J$ with

$$\phi(x, t) = \sum_{i=1}^{n} \phi_i(h_i(x), t) \mathbb{1}_{\omega_i}(x),$$

where $\mathbb{1}_{\omega_i}$ denotes the indicator function of ω_i, with the convention that if $x \in \bigcup_{i \in I} \omega_i$ with $I \subset \{1, \ldots, n\}$, then $\phi(x, t) = \frac{1}{|I|} \sum_{i \in I} \phi_i(h_i(x), t)$. In the rest of the paper, we will identify each function ϕ defined on $\Omega \times J$, with the corresponding function $(\phi_1, \ldots, \phi_n)$ defined on $\omega \times J$.

Finally, the boundary of ω_i can be split into two parts such that

$$\partial\omega_i = \Gamma_i^D \cup \Gamma_i^N, \quad \Gamma_i^D \cap \Gamma_i^N = \varnothing, \quad 1 \leq i \leq n, \tag{18}$$

and that the homeomorphisms h_i, $1 \leq i \leq n$, preserve this cutting, that is

$$\gamma_i^D = h_i(\Gamma_i^D), \quad \gamma_i^N = h_i(\Gamma_i^N), \quad \partial\omega = \gamma_i^D \cup \gamma_i^N, \quad \gamma_i^D \cap \gamma_i^N = \varnothing, \quad 1 \leq i \leq n. \tag{19}$$

Now we consider the following complex network of reaction-diffusion systems

$$\begin{cases} \dfrac{\partial v_i}{\partial t}(\xi, t) = r\Delta v_i(\xi, t) + \delta_i\Big(v_i(\xi, t), \{v_j(\xi, t)\}_{j \in \mathcal{N}_i}\Big) + f_i(\xi, t) & (\xi, t) \in \omega \times \mathbb{R}_+^*, \\ \dfrac{\partial v_i}{\partial \nu}(\xi, t) = 0 & (\xi, t) \in \gamma_i^N \times \mathbb{R}_+^*, \\ v_i(\xi, t) = 0 & (\xi, t) \in \gamma_i^D \times \mathbb{R}_+^*, \\ v_i(\xi, 0) = v_{i,0}(\xi) & \xi \in \omega, \end{cases} \tag{20}$$

for all $i \in \{1, \ldots, n\}$, where ν denotes the outer normal of $\partial\omega$ and $u_0 = (u_{i,0})_{1\leq i\leq n}$ is an initial condition.

Let us finally precise the form of the coupling operator $\delta = (\delta_1, \ldots, \delta_n)$. We suppose that there exists a symmetric matrix of order n denoted by $L = (L_{ij})$, such that

$$L_{ji} > 0, \quad j \in \mathcal{N}_i, \tag{21}$$

and

$$L_{ii} \leq -\sum_{j\in\mathcal{N}_i} L_{ji} < 0, \tag{22}$$

and we set

$$\delta_i\Big(v_i, \{v_j\}_{j\in\mathcal{N}_i}\Big) = L_{ii}v_i + \sum_{j\in\mathcal{N}_i} L_{ji}v_j.$$

4.2 Abstract formulation of the complex network problem

Next, we give an abstract formulation of the complex network problem (20) in order to establish existence and uniqueness results via semigroups techniques. Denote by γ the trace operator that associates to any sufficiently regular function ψ defined on ω its trace defined on $\partial\omega$. For each $i \in \{1, \ldots, n\}$, we introduce the space

$$\mathring{H}^1_{D,i}(\omega) = \{\psi \in H^1(\omega) \; ; \; \gamma\psi = 0 \text{ on } \gamma_i^D\}, \tag{23}$$

whose dual space is $H_D^{-1}(\omega)$. We then consider the triplet of Hilbert spaces $Z \subset Y \subset Z^*$ where

$$Z = \prod_{i=1}^n \mathring{H}^1_{D,i}(\omega), \quad Y = \left(L^2(\omega)\right)^n, \quad Z^* = \left(H_D^{-1}(\omega)\right)^n,$$

and the diagonal operator B defined by

$$B = \operatorname{diag}(B_1, \ldots, B_n),$$

where B_i is the operator defined in $H_D^{-1}(\omega)$, associated to the sesquilinear form given by

$$a_i(u, v) = r\sum_{k=1}^n \int_\omega \frac{\partial u}{\partial \xi_k}\frac{\partial \bar{v}}{\partial \xi_k} d\xi - L_{ii}\int_\omega u\bar{v}d\xi, \quad u, v \in \mathring{H}^1_{D,i}(\omega). \tag{24}$$

In this way, B_i is the realization of the operator $-r\Delta - L_{ii}$ in $H_D^{-1}(\omega)$, under the splitting boundary conditions (Dirichlet boundary condition on γ_i^D and Neumann boundary condition on γ_i^N). It is known that the operators B_i, $1 \leq i \leq n$, and their parts in $L^2(\omega)$ and $\mathring{H}^1_{D,i}(\omega)$, are sectorial operators of $H_D^{-1}(\omega)$, $L^2(\omega)$ and $\mathring{H}^1_{D,i}(\omega)$ respectively, with angles strictly lesser than $\frac{\pi}{2}$, see [49], Theorem 2.5.

Thus the diagonal operator B defined in Z^*, and its parts in Y and Z are sectorial operators of Z^*, Y and Z respectively, with angles strictly lesser than $\frac{\pi}{2}$. Furthermore, it is known that $B_{|Y}$ is a positive and self-adjoint operator in Y, and that the domain of $B_{|Y}$ is such that

$$\begin{aligned} &\mathcal{D}(B_{|Y}) \subset \left(W^{1,p_0}(\omega)\right)^n, \quad p_0 > 2, \\ &\|v\|_{\left(W^{1,p_0}(\omega)\right)^n} \leq C\left(\left\|B_{|Y}v\right\|_Y + \|v\|_Y\right), \quad v \in \mathcal{D}(B_{|Y}), \end{aligned} \tag{25}$$

if the following assumptions are satisfied:

$$\left|B(\xi_0,\,\rho)\cap\gamma_i^D\right|\geq C\rho^{N-1},\quad \xi_0\in\gamma_i^D,$$
$$\left|B(\xi_0,\,\rho)\cap\gamma_i^N\right|\geq C\rho^{N-1},\quad B(\xi_0,\,\rho)\cap\gamma_i^D=\varnothing,\quad \xi_0\in\gamma_i^N,$$

with positive constants ρ, C, where $B(\xi,\,\rho)$ denotes the open ball in $\mathbb{R}^N$ with center $\xi\in\partial\omega$ and radius ρ, see [4], Theorem 2.2.

The operator B admits fractional powers which are defined my means of the Dunford-Riesz integral, whose domains are characterized by the interpolation spaces, see [49], Theorem 2.35, as follows

$$\mathcal{D}(B^\theta)=(Z,\,Y)_{2\theta-1},\quad \tfrac{1}{2}\leq\theta\leq 1. \tag{26}$$

Note that the interpolation space $(Z,\,Y)_{2\theta-1}$ is defined for instance in [48], Theorem 1.9.3. Consequently, we have

$$\mathcal{D}(B^{\frac{1}{2}})=Y,\quad \mathcal{D}(B^\theta)\subset\left(H^{2\theta-1}(\omega)\right)^n,\quad \tfrac{1}{2}<\theta\leq 1. \tag{27}$$

Finally, we suppose that $p_0>N$ and we define the non-linear operator F by setting

$$F(v)=\left(\sum_{j\in\mathcal{N}_i}L_{ji}v_j,\,\ldots,\,\sum_{j\in\mathcal{N}_n}L_{ji}v_j\right)^T, \tag{28}$$

for all $v\in\mathcal{D}(F)=\left(H^{\frac{N}{p_0}}(\omega)\right)^n$.

The abstract formulation of the complex network problem (20) now reads

$$\begin{cases}v'(t)+Bv(t)=F(v(t))+f(t), & t\in(0,\,T],\\ v(0)=u_0.\end{cases} \tag{29}$$

The latter equation is seen to belong to the class of semilinear parabolic abstract equations. In the next section, we show that this problem admits a unique local solution.

4.3 Local solutions for the complex network problem

Let us introduce $\eta=\frac{1}{2}+\frac{N}{2p_0}$, where $p_0>N$ satisfies equation (25) above. By virtue of (27), we have

$$\mathcal{D}(B^\eta)\subset\left(H^{2\eta-1}(\omega)\right)^n.$$

Since $2\eta-1=\frac{N}{p_0}$, it holds that $\mathcal{D}(B^\eta)\subset\mathcal{D}(F)$.

For two exponents β and σ such that $0<\sigma<\beta\leq 1$, the space of weighted Hölder continuous functions defined on $(0,\,T]$, with values in Z^*, is denoted by $\mathcal{F}^{\beta,\sigma}((0,\,T],\,Z^*)$, see [49], section 1.2.4.

Theorem 4.1. *For any initial condition $u_0\in Y$, and any function $f\in\mathcal{F}^{\frac{1}{2},\sigma}((0,\,T],\,Z^*)$ with exponent σ such that $0<\sigma<1-\eta$, the complex network problem (29) admits a unique solution v in the function space*

$$\mathscr{C}((0,\,T_{u_0,f}],\,\mathcal{D}(B))\cap\mathscr{C}([0,\,T_{u_0,f}],\,Y)\cap\mathscr{C}^1((0,\,T_{u_0,f}],\,Z^*), \tag{30}$$

where $T_{u_0,f}$ is a positive final time depending on u_0 and f. Furthermore, v is necessarily given by the representation formula

$$v(t)=v(0)e^{-tB}+\int_0^t e^{-(t-s)B}\left[F(v(s))+f(s)\right]ds,\quad 0\leq t\leq T_{u_0,f}. \tag{31}$$

Proof. As noticed previously, we have $\mathcal{D}(B^\eta) \subset \mathcal{D}(F)$. We claim that

$$\|F(\phi) - F(\psi)\|_{Z^*} \leq \|B^\eta(\phi - \psi)\|_{Z^*} ,$$

for all ϕ, ψ in $\mathcal{D}(B^\eta)$. To prove this assertion, let us consider $\phi = (\phi_1, \dots, \phi_n)$, $\psi = (\psi_1, \dots, \psi_n)$ in $\mathcal{D}(B^\eta)$. For each $i \in \{1, \dots, n\}$, we have, by virtue of the definition of the dual norm:

$$\begin{aligned} \|\phi_i - \psi_i\|_{H_D^{-1}(\omega)} &= \sup_{\|z\|_{\mathring{H}^1_{D,i}} \leq 1} \left| \int_\omega (\phi_i - \psi_i) z d\xi \right| \\ &\leq C \sup_{\|z\|_{\mathring{H}^1_{D,i}} \leq 1} \|\phi_i - \psi_i\|_{L^2(\omega)} \times \|z\|_{L^2(\omega)} , \end{aligned}$$

thanks to the Hölder inequality, where C denotes a positive constant. The Sobolev embeddings, see [49], Theorem 1.36, guaranty that

$$H^1(\omega) \subset L^2(\omega), \quad H^{\frac{N}{p_0}}(\omega) \subset L^2(\omega),$$

which leads to

$$\begin{aligned} \|\phi_i - \psi_i\|_{H_D^{-1}(\omega)} &\leq C \|\phi_i - \psi_i\|_{L^2(\omega)} \\ &\leq C \|\phi_i - \psi_i\|_{H^{\frac{N}{p_0}}(\omega)} , \end{aligned}$$

from which we deduce

$$\|\phi - \psi\|_{Z^*} \leq C \|B^\eta(\phi - \psi)\|_{Z^*} .$$

The conclusion follows from Theorem 4.1 (with $\beta = \frac{1}{2}$) in [49]. □

Remark 4.2. It is possible to apply the latter proof to a more general class of couplings, defined by the superposition of linear couplings and quadratic couplings. However, we focus on this paper on linear couplings, which we will choose in order to guaranty the convergence of the solution of the complex network (20) towards the solution of the generalized diffusion problem (1).

4.4 Global existence and asymptotic behavior

In this section, we investigate sufficient conditions for which the local solutions of the complex network problem (20) are global in time. Sufficient conditions for the solutions of reaction-diffusion equations or systems to be global are well-known for various boundary conditions, such as Dirichlet, Neumann or Robin boundary conditions. It is also observed that the solutions of such equations can explode in finite time if the non-linearities present an over-polynomial growth, see for instance [34] or [38]. However, since the complex network problem (20) involves mixed boundary conditions, we propose another method to prove that the solutions of problem (20) are global in time. Thus we first prove the non-negativity of the solutions stemming from non-negative initial data, under the assumption that the source term f is itself non-negative and we establish an estimation of the solution in L^2 thanks to Poincaré inequality.

Lemma 4.3. *Let $u_0 \in Y$, and $f \in \mathcal{F}^{\frac{1}{2},\sigma}((0, T], Z^*)$ with $0 < \sigma < 1 - \eta$, be such that $u_{0,i}(\xi) \geq 0$, $f_i(\xi, t) \geq 0$ for all $\xi \in \omega$, $t \in (0, T]$ and $1 \leq i \leq n$. Denote by $v = (v_i)_{1 \leq i \leq n}$ the unique solution of problem (29) stemming from u_0, in function space (30). Then we have $v_i(\xi, t) \geq 0$ for all $\xi \in \omega$, $t \in [0, T_{u_0,f}]$ and $1 \leq i \leq n$.*

Proof. We introduce the auxiliary problem given by

$$\frac{\partial \tilde{v}_i}{\partial t} = r\Delta \tilde{v}_i + L_{ii}\tilde{v}_i + \sum_{j\in\mathcal{N}_i} L_{ji} |\tilde{v}_j| + f_i, \quad 1 \le i \le n,$$

with the same initial condition u_0, and the same boundary condition as in (20). We can apply the same method as in the proof of Theorem 4.1 in order to prove that the auxiliary problem admits a unique local solution $\tilde{v}$ defined on some interval $\left[0,\, \tilde{T}_{u_0,f}\right]$ with $\tilde{T}_{u_0,f} > 0$. Furthermore, we consider the truncation function χ defined on $\mathbb{R}$ by

$$\chi(s) = \begin{cases} 0 \text{ if } s \ge 0, \\ s^2 \text{ else.} \end{cases}$$

It is easy to see that χ is continuously differentiable on $\mathbb{R}$ and satisfies the properties

$$\chi(s) \ge 0, \quad \chi'(s) \le 0, \quad s\chi'(s) \ge 0, \quad s \in \mathbb{R}. \tag{32}$$

Next we define for each $i \in \{1, \ldots, n\}$ the function ρ_i by setting

$$\rho_i(t) = \int_\omega \chi(\tilde{v}_i(\xi, t))d\xi, \quad t \in \left[0,\, \tilde{T}_{u_0,f}\right].$$

We have $\rho_i(t) \ge 0$ for all $t \in \left[0,\, \tilde{T}_{u_0,f}\right]$ by construction and $\rho_i(0) = 0$ since u_0 admits non-negative components. Moreover, ρ_i is continuously differentiable on $\left[0,\, \tilde{T}_{u_0,f}\right]$, with

$$\begin{aligned}
\rho_i'(t) &= \int_\omega \frac{\partial \tilde{v}_i}{\partial t}\chi'(\tilde{v}_i)d\xi \\
&= \int_\omega \left[r\Delta\tilde{v}_i + L_{ii}\tilde{v}_i + \sum_{j\in\mathcal{N}_i} L_{ji}|\tilde{v}_j| + f_i \right] \chi'(\tilde{v}_i)d\xi \\
&= r\int_\omega \Delta\tilde{v}_i\chi'(\tilde{v}_i)d\xi + L_{ii}\int_\omega \tilde{v}_i\chi'(\tilde{v}_i)d\xi + \sum_{j\in\mathcal{N}_i} L_{ji}\int_\omega (|\tilde{v}_j| + f_i)\,\chi'(\tilde{v}_i)d\xi \\
&= -r\int_\omega |\nabla\chi'(\tilde{v}_i)|^2 d\xi + L_{ii}\int_\omega \tilde{v}_i\chi'(\tilde{v}_i)d\xi + \sum_{j\in\mathcal{N}_i} L_{ji}\int_\omega (|\tilde{v}_j| + f_i)\,\chi'(\tilde{v}_i)d\xi,
\end{aligned}$$

where we omit the variables ξ and t in order to lighten our notations. It follows from (22) and (32) that $\rho_i'(t) \le 0$ for all $t \in \left[0,\, \tilde{T}_{u_0,f}\right]$. We can deduce that $\rho_i \equiv 0$ on $\left[0,\, \tilde{T}_{u_0,f}\right]$, which means that $\tilde{v}_i(\xi, t) \ge 0$ for all $\xi \in \omega$, $t \in \left[0,\, \tilde{T}_{u_0,f}\right]$ and $1 \le i \le n$. Thus $\tilde{v}$ is also a solution of the initial complex network problem (29). By uniqueness, we obtain $\tilde{v} = v$ on $[0,\, T_{u_0,f}] \cap \left[0,\, \tilde{T}_{u_0,f}\right]$. Finally, we easily prove that $T_{u_0,f} = \tilde{T}_{u_0,f}$ and this achieves the proof. □

The next proposition establishes an energy estimate of the solutions of the complex network problem (20).

Proposition 4.4. *Assume that $u_0 \in Y$ and $f \in L^\infty(\omega \times [0,\, T])$ satisfy $u_{0,i}(\xi) \ge 0$, $f_i(\xi, t) \ge 0$ for all $\xi \in \omega$, $t \in (0,\, T]$ and $1 \le i \le n$. Let $v = (v_i)_{1\le i\le n}$ denote the unique solution of the complex network problem (20) stemming from u_0, defined on $[0,\, T_{u_0,f}]$. Then we have:*

$$\sum_{i=1}^n \|v_i(t)\|^2_{L^2(\omega)} \le e^{-Kt}\sum_{i=1}^n \|u_{0,i}\|^2_{L^2(\omega)} + \frac{4n\,|\omega|\,\|f\|_{L^\infty(\omega\times[0,T])}}{K^2}, \quad 0 < t \le T_{u_0,f}, \tag{33}$$

for some positive constant K.

Proof. The duality product in $H_D^{-1}(\omega) \times \mathring{H}^1_{D,i}(\omega)$ between the equation giving the state of node v_i in system (20) and v_i leads to

$$\frac{1}{2}\frac{d}{dt}\int_\omega v_i^2(\xi,t)d\xi + r\int_\omega |\nabla v_i(\xi,t)|^2\, d\xi \le \int_\omega f_i(\xi,t)v_i(\xi,t)d\xi, \quad 0 < t \le T_{u_0,f}.$$

By virtue of Poincaré inequality 2.3, we have

$$\|v_i\|_{L^2(\omega)} \le C_i \|\nabla v_i\|_{L^2(\omega)},$$

for some positive constants C_i, $1 \le i \le n$, since $v_i \in \mathring{H}^1_{D,i}(\omega)$. It follows that

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^n\int_\omega v_i^2(\xi,t)d\xi + K\sum_{i=1}^n\int_\omega v_i^2(\xi,t)d\xi \le \sum_{i=1}^n\int_\omega f_i(\xi,t)v_i(\xi,t)d\xi, \quad 0 < t \le T_{u_0,f},$$

for some positive constant $K > 0$. By assumption, we have $f \in L^\infty(\omega \times [0,\, T])$, from which we deduce

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^n\int_\omega v_i^2(\xi,t)d\xi + K\sum_{i=1}^n\int_\omega v_i^2(\xi,t)d\xi \le \|f\|_{L^\infty(\omega\times[0,\,T])}\sum_{i=1}^n\int_\omega v_i(\xi,t)d\xi, \quad 0 < t \le T_{u_0,f}.$$

Next, we use the elementary inequality

$$s \le \theta s^2 + \frac{1}{\theta}, \quad \forall s \in \mathbb{R},$$

with $\theta > 0$ arbitrarily small. We obtain

$$\|f\|_{L^\infty(\omega\times[0,\,T])}\sum_{i=1}^n\int_\omega v_i(\xi,t)d\xi \le \theta\,\|f\|_{L^\infty(\omega\times[0,\,T])}\sum_{i=1}^n\int_\omega v_i^2(\xi,t)d\xi + \frac{n\,|\omega|}{\theta},$$

which leads to

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^n\int_\omega v_i^2(\xi,t)d\xi + \frac{K}{2}\sum_{i=1}^n\int_\omega v_i^2(\xi,t)d\xi \le \frac{2n\,|\omega|\,\|f\|_{L^\infty(\omega\times[0,\,T])}}{K},$$

by choosing $\theta = \frac{K}{2\|f\|_{L^\infty(\omega\times[0,\,T])}}$. Applying Gronwall lemma 2.2 yields the conclusion. □

As a first corollary of the latter proposition, we directly obtain the global in time existence of the solutions of the complex network problem (20).

Corollary 4.5. *Under the assumptions of proposition 4.4, the solution v of the complex network (20) stemming from u_0 is global, that is $T_{u_0,f} = T$. Furthermore, in the case where $f \in L^\infty(\Omega \times (0,\, +\infty))$, then $T = +\infty$.*

The energy estimate (33) also guarantees that the complex network problem (29) determines a continuous dynamical system $(S(t),\, \Phi,\, Y)$ defined in Y, with phase space

$$\Phi = \{u_0 \in Y \;;\; u_{0,i}(\xi) \ge 0 \text{ in } \omega,\; 1 \le i \le n\}, \tag{34}$$

where $S(t)$ is the semigroup of non-linear operators defined by

$$\begin{array}{rlcl} S(t): & \Phi \times (0,\, +\infty) & \longrightarrow & \Phi \\ & (u_0,\, t) & \longmapsto & v(t,\, u_0), \end{array}$$

where $v(t,\, u_0)$ denotes the unique solution of problem (20) stemming from u_0. By virtue of Theorem 6.15 in [49], we have the following corollary.

Corollary 4.6. *The continuous dynamical system $(S(t),\, \Phi,\, Y)$ defined above admits a family $(\mathcal{M})$ of exponential attractors.*

We recall that an exponential attractor is a compact subset $\mathfrak{M} \subset \Phi$ which is positively invariant and attracts the bounded subsets of Φ at an exponential rate.

5 Sufficient conditions of convergence

In this section, our aim is to establish sufficient conditions for the solution of the complex network problem (20) to converge to the solution of the generalized diffusion equation (1), when the number n of nodes in the network tends to infinity. Our approach consists in constructing an intermediate finite differences scheme which links the solution of the generalized diffusion problem (1) to the solution of the complex network (20).

5.1 Consistency and stability assumptions

First of all, we define a double discretization in space and time of Ω and $[0, T]$, where T is a given positive final time. For each $i \in \{1, \ldots, n\}$, we choose $\bar{x}_i \in \omega_i$, and we set

$$\delta x = \max_{1 \leq i \leq n} \sup_{y, z \in \omega_i} \|y - z\|_{\mathbb{R}^d} .$$

The coefficient δx is regarded as the size of the space discretization. We also introduce a discretization of the time interval $[0, T]$ of step $\delta t > 0$:

$$0 < \delta t < 2\delta t < \cdots < k\delta t < \cdots < m\delta t, \quad 0 \leq k \leq m, \tag{35}$$

where m is a positive integer. For each function ϕ defined in $\Omega \times [0, T]$, we set

$$\begin{aligned} \phi_i^k &= \phi(\bar{x}_i, k\delta t), \quad 1 \leq i \leq n, \quad 0 \leq k \leq m, \\ \phi^k &= (\phi_1^k, \ldots, \phi_n^k)^T, \quad 0 \leq k \leq m. \end{aligned} \tag{36}$$

Additionally, we assume that one can choose two matrices Q_1 and Q_2 of order n, such that the following explicit finite differences scheme

$$\frac{w^{k+1} - w^k}{\delta t} = Q_1 w^k + f^k, \tag{37}$$

approximates the heat equation

$$\frac{\partial y}{\partial t} = r\Delta y + f, \tag{38}$$

while the second following scheme

$$\frac{w^{k+1} - w^k}{\delta t} = Q_2 w^k, \tag{39}$$

approximates the biharmonic equation

$$\frac{\partial z}{\partial t} = -\Delta^2 z. \tag{40}$$

More precisely, we assume that there exists a norm $\|\cdot\|_\star$ in $\mathbb{R}^n$ such that the following assumptions are satisfied.

- (H_1) [consistency of scheme (37)]: the first scheme (37) is consistent for the norm $\|\cdot\|_\star$, with accuracy $d_1 > 0$ in space and $e_1 > 0$ in time, that is

  $$\frac{y^{k+1} - y^k}{\delta t} = Q_1 y^k + f^k + \tau_1, \quad \|\tau_1\|_\star = O((\delta x)^{d_1}) + O((\delta t)^{e_1}),$$

  where y denotes a solution of the heat equation (38) in Ω with fixed boundary condition.

- (H_2) [consistency of scheme (39)]: the second scheme (39) is consistent for the norm $\|\cdot\|_\star$, with accuracy $d_2 > 0$ in space and $e_2 > 0$ in time, that is

$$\frac{z^{k+1} - z^k}{\delta t} = Q_2 z^k + \tau_2, \quad \|\tau_2\|_\star = O((\delta x)^{d_2}) + O((\delta t)^{e_2}),$$

where z denotes a solution of the biharmonic equation (40) in Ω with fixed boundary condition.

- (H_3) [conditional stability of schemes (37) and (39)]: the matrices Q_1 and Q_2 satisfy

$$\|Q_i u_0\|_\star \leq K_i \|u_0\|_\star, \quad \forall n \geq 0, \quad \forall u_0 \in \mathbb{R}^n, \quad i \in \{1, 2\},$$

with positive constants K_1 and K_2, provided δx and δt fulfill a common stability condition of the type

$$h(\delta x, \delta t) \leq 0,$$

where h denotes a polynomial of degree 2.

Before we state our convergence result, we show that the above assumptions can easily be satisfied at least for space dimensions $d = 1$ and $d = 2$ (see for instance [50] and references therein cited for the three-dimensional case).

First, suppose that $d = 1$ and $\Omega = (0, 1)$. The discretization of Ω can be defined by setting $\delta x = \frac{1}{n}$ and $\omega_i = (i\delta x, (i + 1)\delta x)$ for $0 \leq i \leq n - 1$. It is known, see for instance [5] or [45], that scheme (37) can be defined stemming from the expression

$$\Delta u(x) = \frac{u(x - \delta x) - 2u(x) + u(x + \delta x)}{(\delta x)^2} + O((\delta x)^2),$$

which determines the *stencil* of the scheme:

$$\frac{1}{(\delta x)^2}\left[\quad 1 \quad\quad -2 \quad\quad 1 \quad\right].$$

In parallel, scheme (39) can be determined by the 5 points stencil

$$\frac{1}{(\delta x)^2}\left[\quad -1 \quad\quad 4 \quad\quad -6 \quad\quad 4 \quad\quad -1 \quad\right].$$

In this way, it is verified that schemes (37) and (39) are consistent for the L^2 norm defined by

$$\|u\|_\star = \sqrt{\sum_{j=1}^{n} |u_j|^2 \, \delta x}, \quad u \in \mathbb{R}^n,$$

and that those schemes are conditionally stable in this norm, under the Courant-Friedrichs-Lewy condition (CFL)

$$\delta t - K(\delta x)^2 \leq 0,$$

for some positive constant K.

Now suppose that the space dimension is $d = 2$, and assume that $\Omega = (0, 1)^2$. The discretization of Ω can be defined by setting $\delta x = \frac{1}{\sqrt{n}}$ and

$$\overline{\Omega} = \bigcup_{1 \leq i_1, i_2 \leq \sqrt{n}} \overline{\omega}_{i_1, i_2}, \quad \omega_{i_1,i_2} = (i_1\delta x, (i_1 + 1)\delta x) \times (i_2\delta x, (i_2 + 1)\delta x), \quad 1 \leq i_1, i_2 \leq \sqrt{n},$$

provided n is chosen as a squared integer. Scheme (37) can be defined by the 5 points stencil

$$\frac{1}{(\delta x)^2}\begin{bmatrix} & 1 & \\ 1 & \text{-4} & 1 \\ & 1 & \end{bmatrix},$$

whereas scheme (39) can be defined by the 13 points stencil

$$\frac{1}{(\delta x)^2}\begin{bmatrix} & & \text{-1} & & \\ & \text{-2} & 8 & \text{-2} & \\ \text{-1} & 8 & \text{-20} & 8 & \text{-1} \\ & \text{-2} & 8 & \text{-2} & \\ & & \text{-1} & & \end{bmatrix}.$$

Once again, it is known that those expressions determine L^2 consistent schemes, which are conditionally stable in the L^2 norm under the CFL condition.

Remark 5.1. It is worth emphasizing that the stencils corresponding to the approximation of the biharmonic equation (40) involve a greater number of points than the stencils corresponding to the approximation of the heat equation (38), thus we recover the principle of diffusion "in the neighborhood of the neighborhood" mentioned in our introduction.

Furthermore, it is observed that the central coefficients of the stencils presented above are negative and that the sums of the coefficients of the stencils always equal zero. Consequently, we can choose to define the couplings of the complex network problem (20) by setting

$$L = Q_2. \tag{41}$$

In this way, we easily verify that conditions (21) and (22) hold for matrix L.

Obviously, many other approximation schemes could be considered equivalently, see [45], but we are not focusing on this point in this paper.

5.2 Convergence result

Here, we finally establish the convergence of the solution of the complex network problem (20) towards the solution of the generalized diffusion problem (1), under sufficient conditions which guarantee a relevant choice of the connectivity matrix L.

Proposition 5.2. *Assume that hypotheses (H_1), (H_2) and (H_3) are fulfilled, and furthermore, that the connectivity matrix L is defined by (41). Let f be a source term in $L^\infty(\Omega \times [0, T])$, u_0 an initial condition in $(\mathcal{D}(\mathcal{A}), X)_{\frac{1}{2},2} \cap \Phi$, respectively given by (16) and (34). Denote by u and v the global solutions of problems (1) and (20) respectively. Then, it holds that*

$$\sup_{0\le k\le m} \left\| u^k - v^k \right\|_\star = O\big((\delta x)^{d_3}\big) + O\big((\delta t)^{e_3}\big),$$

with $d_3 = \max(d_1, d_2)$ and $e_3 = \max(e_1, e_2)$.

Proof. Let us introduce the following approximation scheme

$$\frac{w^{k+1} - w^k}{\delta t} = (Q_1 + L)w^k + f^k, \quad 0 \le k \le m. \tag{42}$$

First, we show that the solution v of the complex network problem (20) converges to w. To that aim, we compute for each $k \in \{0, \dots, m\}$

$$\frac{v^{k+1} - v^k}{\delta t} - \frac{w^{k+1} - w^k}{\delta t} = Q_1 v^k + Lv^k + f^k + \tau_1 - \big[(Q_1 + L)w^k + f^k\big] = \tau_1,$$

from which we can deduce that

$$v^{k+1} - w^{k+1} = [I_n + \delta t(Q_1 + L)](v^k - w^k).$$

We can deduce from the conditional stability of matrices Q_1 and L, which holds by virtue of assumption (H_3), that matrix $Q_1 + L$ is also conditionally stable. Using classical methods of Lax theorem, see [45], it follows that v converges towards w, with accuracy d_1 in space and e_1 in time.

Using similar arguments, we show that the solution u of the generalized diffusion problem (1) also converges to w, with accuracy d_2 in space and e_2 in time.

The conclusion follows from the triangular inequality. □

As a direct consequence, we obtain the following corollary.

Corollary 5.3. *Each solution of the generalized diffusion problem* (1) *is attracted at an exponential rate by the attractors* $(\mathcal{M})$ *of the complex network problem.*

Conclusion and perspectives

In this paper, we have analyzed a generalized diffusion problem modeling distant diffusion in population dynamics. Using sectorial operators and analytic semigroups techniques, we have proved new existence and regularity results. Additionally, we have proposed a new approximation of our generalized diffusion problem by a complex network of reaction-diffusion equations. In a restricted set of reasonable assumptions, we have obtained a first convergence result, from which we have deduced the existence of exponential attractors for the generalized diffusion problem.

In a future work, we aim to enlarge our framework, by considering other boundary conditions for the initial generalized diffusion problem and by adding non-linearities in the source term. Furthermore, we believe that the convergence results can also be extended to a more general functional spaces setting.

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