Analytic semigroup generated by the dispersal process of a sylvatic transmission model of Chagas disease

Narimene Benarbia, Rabah Labbas, Tewfik Mahdjoub, Alexandre Thorel

N. B. & T. M.: NLAAML, Abou-Bekr Belkaïd University, Tlemcen, 13000, Algeria
narimene.benarbia@univ-tlemcen.dz, tewfik.mahdjoub@univ-tlemcen.dz
R. L. & A. T.: Université Le Havre Normandie, Normandie Univ, LMAH UR 3821, 76600 Le Havre, France.
rabah.labbas@univ-lehavre.fr, alexandre.thorel@univ-lehavre.fr

Abstract

In this work, we develop a new biological transmission model for Chagas disease. This model, set in two juxtaposed habitats with skew Brownian motion conditions at the interface, is composed of two reaction-diffusion equations and takes into account the sylvatic transmission. We write it as an abstract perturbed Cauchy problem using operator theory. Then, we show that the main operator, which models the dispersal process, generates an analytic semigroup in an adequate Banach space.
Key Words and Phrases: Chagas disease, reaction-diffusion equations, skew Brownian motion, sylvatic transmission, analytic semigroup.
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1 Introduction

According to World Health Organization (WHO) statistics, between six and seven million people of the world population are affected by Chagas disease see [32] and [34]. The disease, endemic in Latin America with a morbidity of 12 000 deaths/year, has seen its geographical expansion reach the countries of North America, Europe, Australia and Asia [28]. On the African continent, the disease is present, for instance, in Gabon and Tanzania with less than 900 cases/country.

Chagas disease or American trypanosomiasis is a vector-borne disease caused by the flagellate protozoan parasite *Trypanosoma cruzi* (*T. cruzi*). This parasite can infect a wide range of mammalian hosts (including humans) or domestic or wild birds. The main mode of transmission is contact with insect vectors called triatominae. When the latter live in shelters close to human dwellings, the transmission of the parasite is said to be domestic. If they live in the nests of mammals or undomesticated birds, the transmission is sylvatic. Some vector species such as *T. Dimidiata* have adapted to both domestic and sylvatic habitats: transmission in this case is peridomestic.

During the typical infection cycle, called the kissing bug, an infected triatomine takes a host's blood meal and releases the parasite in their feces near the bite. The parasite enters the host through the wound or an intact mucous membrane, such as the conjunctiva: this is called sterocorary transmission. An infected host also transmits the parasite to a healthy vector during blood meals and the parasite then resides in the intestine of the vector. Transmission in hosts can also occur congenitally, (vertical transmission from infected mother to child) and orally.

Vector control remained the main strategy to counter the spread of the disease. It was based on the regular spraying of insecticide in the affected villages. This fight, carried out jointly by several Latin American countries within the framework of intergovernmental projects, cost US$ 30 million annually without being able to achieve the objectives set by the WHO. Indeed, areas where the presence of the vector has been drastically reduced, have seen the installation of a process of re-infestation by triatomines [2].

The objective of mathematical modeling is therefore to try to explain this phenomenon and then to determine the main demographic parameters that make a triatomine invasion successful in order to allow decision-makers to control these parameters. It was based on various mathematical tools selected according to the established objectives. To demonstrate that, in the sylvatic (wild) case, oral contamination is as significant as classical transmission via bites during a blood meal, [21] proposed an SI model based on ordinary differential equations (ODE). This model incorporates a prey-predator sub-model where the vector seeks a host for a blood meal, while the host -generally insectivorous- feeds on the vector. Model analysis shows that if the vector-host contact rate rapidly reaches saturation (the host reaches satiety quickly), then the vector population exhibits two stable equilibrium states: one at high vector density and the other at low vector density. Controlling the infection by shifting it from high to low density involves crossing a critical threshold. If this threshold is not reached, the population returns to its equilibrium state. The drawback of this model is that it does not take the spatial component into account.

The geographical habitat of vectors was considered in the model proposed by [27]. This work follows a "population ecology" approach and places the vector population in a scenario of invading a pristine sylvatic space. This biological invasion process is modeled using traveling waves, which are solutions to a system of integro-difference equations. Using this tool, the invasion speed representing the displacement velocity of traveling waves moving from colonized to non-colonized space is estimated for the vector population. However, the model cannot conclude on the invasion speed of *T. cruzi*, which is not necessarily the same as its vector's invasion speed. Furthermore, this model assumes that once a habitat point is colonized, it cannot be decolonized. This implies a certain advection of the vector flux, which is not observed in the field.

Still aiming to estimate the spread speed of the *T. cruzi* parasite in the sylvatic case, practical studies have been developed. A first cellular automaton (CA) model, constructed by [8], divides the geographic area between Mexico and the southern United States into a grid of 9,376 contiguous cells. In each cell, two systems of differential equations are written, integrating the contacts of two vector/host pairs. These ODE systems, solved numerically, allow for the deduction of *T. cruzi* prevalence in each cell. On the grid, prevalences exceeding 7% define infected cells, enabling the deduction of the disease's invasion speed. Although Kribs' CA provides invasion speeds depending on the relief of the patch, the model does not explicitly inform decision-makers on which factor to act -vector or hosts- to stop the infection.

To answer this latter question, [10] developed an agent-based model (multi-agent model) accounting for more complex interactions between hosts and vectors. Several infection inoculation scenarios on a 100x100 cell grid were monitored, and numerical simulations show that the infection is initially carried by vectors and then persists thanks to the hosts. Consequently, disease control should not be limited to insecticide use but must involve action on the hosts, especially since the model shows that vertical transmission has a significant impact on maintaining high prevalence.

In the context of peridomestic transmission, a model based on reaction-diffusion equations was proposed by [9]. This type of partial differential equation (PDE) integrates both spatial dispersion processes and the demographics of vectors and hosts. The model defines a buffer zone between the village and the source of infection (the forest), where the *T. cruzi* parasite is present following Brownian motion. Each habitat point is not permanently colonized, which more closely reflects field reality. The authors demonstrate that the problem is mathematically well-posed, providing the opportunity for subsequent numerical exploitation.

In the present work, we maintain the principle of the existence of Brownian motion of the parasite on either side of the boundary between the infected zone and the healthy zone in the case of sylvatic transmission. Vertical transmission of the parasite in hosts is also integrated into the reaction-diffusion equations.

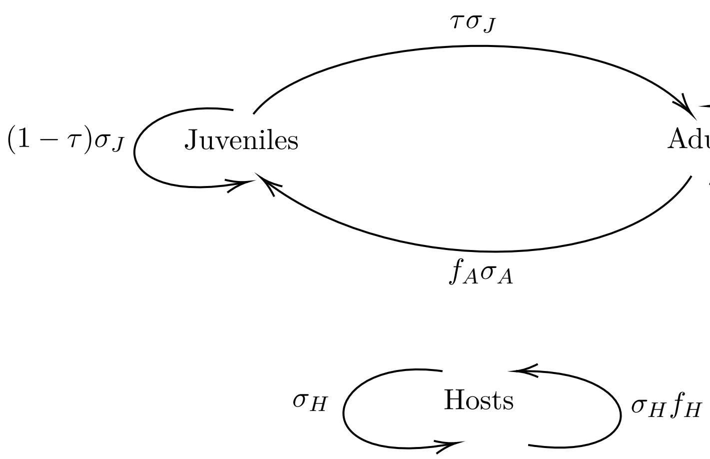
Sylvatic transmission is considered over a domain $\Omega = [-\ell, L] \times [0,1] \subset \mathbb{R}^2$, representing the natural habitat of the vectors and the hosts, considered in two health states: susceptible and infected. The susceptible individuals live on $\Omega_S = [0, L] \times [0,1]$ while the infected one are on $\Omega_I = [-\ell, 0] \times [0,1]$. Thus, the spatio-temporal dynamics of vector and host populations induce skew Brownian motion conditions of individuals across the common border $\Gamma = \{0\} \times [0,1]$, see equations (1) below. On the other hand, the demographic and spatial dispersal processes, on the juxtaposed subdomains Ω_S and Ω_I are described by reaction-diffusion equations, see (3) below. The system obtained is then written, in an adequate functional space, in the form of an abstract differential equation.

Then, our aim is to show that the principal dispersal operator generates an analytic semigroup.

2 The model

2.1 Demography and disease transmission

The life cycle of the Triatominae consists of seven stages of development: an egg stage, five larval stages and an adult stage. Following [26], the development from the egg to the fifth larval stage is considered as a single stage called the juvenile stage. Let denote respectively by $J(t,x,y)$, $A(t,x,y)$ and $H(t,x,y)$, the densities of juvenile vectors, adult vectors and hosts, at time t at point (x,y). Two health states are considered: susceptible individuals, non-carriers of the *T.Cruzi* parasite, of densities J_S, A_S and H_S and infected individuals, carriers of the parasite, of densities J_I, A_I and H_I.
Between t and $t+dt$, the juveniles having survived until t with a probability σ_J will pass to the adult stage with a probability τ_J or will remain juveniles with a probability $(1-\tau_J)$. Adults that survived with probability σ_A will lay eggs with fecundity rate f_A. The host mammals have a survival rate σ_H and a fecundity f_H. For simplicity, we will consider that all these demographic parameters are constant. Their interval are given in Table 1. The life cycles of vectors and hosts are schematized in Figure 2.1 below.



The disease is not inherited in vectors, *i.e.* infected adult triatomines will give susceptible juveniles. However, it has been found that in mammalian hosts there is a certain vertical transmission rate which we will denote by ν, see [25]. This will be the only process assumed to be generated by parental effects.

When a juvenile vector, respectively adult, is contaminated by an infected host during the blood meal, the transmission rate of *T. cruzi* is noted Λ_J, respectively Λ_A. A susceptible host becomes infected via juvenile or adult vectors with a rate Λ_H. Generally, these transmission parameters are density-dependent and are written from the WAIFW matrix (Who Acquires Infection From Whom), see [20]. For reasons of simplification, we will assume that these rates are constant in the interval $]0,1[$, see Table 1 below.

2.2 Habitat of vectors and their hosts

The habitat of vectors and their hosts is a part of the forest represented by $\Omega = [-\ell, L] \times [0,1]$. It is divided into two juxtaposed subdomains, $\Omega_S = [0,L] \times [0,\ 1]$ where susceptible individuals live and $\Omega_I = [-\ell, 0] \times [0,1]$ where infected individuals live. The common boundary is $\Gamma = \{0\} \times [0,1]$.

On Ω, juvenile and adult vectors and their hosts diffuse respectively with constant diffusion coefficients $d_J > 0$, $d_A > 0$ and $d_H > 0$.

More precisely, we denote d_J^+, d_A^+ and d_H^+ the diffusion coefficients on Ω_S and d_J^-, d_A^- et d_H^- the diffusion coefficients on Ω_I. The same notation will be adopted for the demographic coefficients.

Table 1 below contains the demographic diffusion and disease transmission coefficients defined on Ω_S and Ω_I.

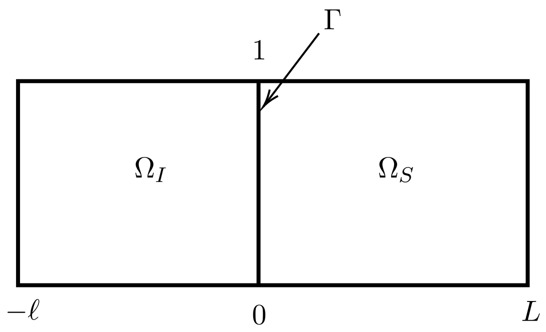


Figure 1: Habitat of vectors and their hosts

Parameters	**Definitions**	**Properties**
σ_J^+, σ_J^-	Probability of survival of juvenile vectors	$0 \leq \sigma_J^+, \sigma_J^- \leq 1$
σ_A^+, σ_A^-	Probability of survival of adult vectors	$0 \leq \sigma_A^+, \sigma_A^- \leq 1$
σ_H^+, σ_H^-	Probability of survival of hosts	$0 \leq \sigma_H^+, \sigma_H^- \leq 1$
τ^+, τ^-	Transition probability from juvenile to adult vectors	$0 \leq \tau^+, \tau^- \leq 1$
f_A^+, f_A^-	Adult vector female fecundity	$f_A^+, f_A^- \geq 0$
f_H^+, f_H^-	Host female fecundity	$f_H^+, f_H^- \geq 0$
ν	Vertical transmission rate per host female	$\nu \geq 0$
Λ_J	Infection rate of juvenile vectors	$\Lambda_J > 0$
Λ_A	Infection rate of adult vectors	$\Lambda_A > 0$
Λ_H	Infection rate of hosts	$\Lambda_H > 0$
d_J^+, d_J^-	Diffusion coefficient of juvenile vectors	$d_J^+, d_J^- > 0$
d_A^+, d_A^-	Diffusion coefficient of adult vectors	$d_A^+, d_A^- > 0$
d_H^+, d_H^-	Diffusion coefficient of hosts	$d_H^+, d_H^- > 0$

Table 1: Definitions and properties of demographic diffusion vertical transmission and disease process

2.3 Skew Brownian motion across the interface Γ

Following [6], it is assumed that there is a flux of juvenile vectors, adult vectors and host that cross the interface Γ, described by the asymmetric equations:

$$\left\{\begin{array}{lcll} p_J\, d_J^- \dfrac{\partial J_I}{\partial x}(t,x,y) & = & (1-p_J)\, d_J^+ \dfrac{\partial J_S}{\partial x}(t,x,y) & \\ p_A\, d_A^- \dfrac{\partial A_I}{\partial x}(t,x,y) & = & (1-p_A)\, d_A^+ \dfrac{\partial A_S}{\partial x}(t,x,y) & t>0,\ (x,y)\in\Gamma \\ p_H\, d_H^- \dfrac{\partial H_I}{\partial x}(t,x,y) & = & (1-p_H)\, d_H^+ \dfrac{\partial H_S}{\partial x}(t,x,y), & \end{array}\right. \tag{1}$$

where p_J, p_A and $p_H \in (0,1)$, are respectively the probabilities that a juvenile vector, an adult vector and a host, being in Ω_I crosses the interface Γ to be in Ω_S.

Demographic, heredity and infection processes generate susceptible individuals who are in the domain Ω_I of infected and infected individuals who are in the domain Ω_S of susceptible.
During the diffusion process, the susceptible join Ω_S and the infected Ω_I. The different flows that cross Γ are written from T_1 to T_5.

Details of the movements are summarized in Table 2 below. We will assume that these are the only moves that take place across the boundary Γ.

Flow	Direction	Description
T_1	$\Omega_I \to \Omega_S$	Susceptible juvenile vectors descended from infected adult vectors
T_2	$\Omega_I \to \Omega_S$	Susceptible hosts descended from infected hosts (no inheritance)
T_3	$\Omega_S \to \Omega_I$	Susceptible hosts that become infected
T_4	$\Omega_S \to \Omega_I$	Susceptible adult vectors that become infected
T_5	$\Omega_S \to \Omega_I$	Susceptible juvenile vectors that become infected

Table 2: Description of the different flows crossing the interface Γ

These flows are known and are defined by:

$$\begin{cases} T_1 &=& f_A^- \, \sigma_A^- \, d_A^- \, \dfrac{\partial A_I}{\partial x} \,|_\Gamma \\ T_2 &=& (1-\nu) \; f_H^- \, \sigma_H^- \, d_H^- \, \dfrac{\partial H_I}{\partial x} \,|_\Gamma \\ T_3 &=& \sigma_H^+ \, d_H^+ \, \dfrac{\partial H_S}{\partial x} \,|_\Gamma + \sigma_H^+ \, f_H^+ \, d_H^+ \, \dfrac{\partial H_S}{\partial x} \,|_\Gamma + (1-\nu) \; f_H^- \, \sigma_H^- \, d_H^- \, \dfrac{\partial H_I}{\partial x} \,|_\Gamma \\ T_4 &=& \tau^+ \, \sigma_J^+ \, d_J^+ \, \dfrac{\partial J_S}{\partial x} \,|_\Gamma + \sigma_A^+ \, d_A^+ \, \dfrac{\partial A_S}{\partial x} \,|_\Gamma \\ T_5 &=& \Big(1-\tau^+\Big) \; \sigma_J^+ \, d_J^+ \, \dfrac{\partial J_S}{\partial x} \,|_\Gamma + f_A^+ \, \sigma_A^+ \, d_A^+ \, \dfrac{\partial A_S}{\partial x} \,|_\Gamma + T_1. \end{cases} \tag{2}$$

These operators T_i, $i = 1,2,3,4,5$, of traces, acting on the interface Γ, will compose the perturbation of the dispersal operator in the reaction-diffusion system written below.

3 Reaction-diffusion system and its operational formulation

The complete reaction-diffusion model considered in this paper is the following

$$\begin{cases} \dfrac{\partial J_I}{\partial t} &=& d_J^- \Delta J_I + \Big(\left(1-\tau^-\right) \, \sigma_J^- - 1\Big) \; J_I + \sigma_A^- \, f_A^- \, A_I + T_5 \, \Lambda_J, & t>0, \; (x,y) \in \Omega_I \\ \dfrac{\partial A_I}{\partial t} &=& d_A^- \Delta A_I + \tau^- \, \sigma_J^- \, J_I + \Big(\sigma_A^- - 1\Big) \; A_I + T_4 \, \Lambda_A, & t>0, \; (x,y) \in \Omega_I \\ \dfrac{\partial H_I}{\partial t} &=& d_H^- \Delta H_I + \Big(\Big(1+\nu \, f_H^-\Big) \; \sigma_H^- - 1\Big) \; H_I + T_3 \, \Lambda_H, & t>0, \; (x,y) \in \Omega_I \\ \dfrac{\partial J_S}{\partial t} &=& d_J^+ \Delta J_S + \Big(\Big(1-\tau^+\Big) \; \sigma_J^+ - 1\Big) \; J_S + f_A^+ \, \sigma_A^+ \, A_S + T_1, & t>0, \; (x,y) \in \Omega_S \\ \dfrac{\partial A_S}{\partial t} &=& d_A^+ \Delta A_S + \tau^+ \, \sigma_J^+ \, J_S + \Big(\sigma_A^+ - 1\Big) \; A_S, & t>0, \; (x,y) \in \Omega_S \\ \dfrac{\partial H_S}{\partial t} &=& d_H^+ \Delta H_S + \Big(\Big(1+f_H^+\Big) \, \sigma_H^+ - 1\Big) \; H_S + T_2, & t>0, \; (x,y) \in \Omega_S, \end{cases} \tag{3}$$

with the initial conditions

$$\begin{cases} J_I(0,x,y) &=& J_I^0(x,y), \\ A_I(0,x,y) &=& A_I^0(x,y), & (x,y) \in \Omega_I \\ H_I(0,x,y) &=& H_I^0(x,y), \end{cases} \qquad \begin{cases} J_S(0,x,y) &=& J_S^0(x,y), \\ A_S(0,x,y) &=& A_S^0(x,y), & (x,y) \in \Omega_S, \\ H_S(0,x,y) &=& H_S^0(x,y), \end{cases} \tag{4}$$

the boundary conditions

$$\begin{cases} (J_I(t,\sigma_1,\sigma_2), A_I(t,\sigma_1,\sigma_2), H_I(t,\sigma_1,\sigma_2)) &=& (0,0,0), & t>0, \; (\sigma_1,\sigma_2) \in \partial\Omega_I \setminus \Gamma \\ (J_S(t,\sigma_1,\sigma_2), A_S(t,\sigma_1,\sigma_2), H_S(t,\sigma_1,\sigma_2)) &=& (0,0,0), & t>0, \; (\sigma_1,\sigma_2) \in \partial\Omega_S \setminus \Gamma, \end{cases} \tag{5}$$

and the transmission conditions

$$
\begin{cases}
J_I(t,0,y) &= J_S(t,0,y), & t>0,\ y\in(0,1)\\
A_I(t,0,y) &= A_S(t,0,y), & t>0,\ y\in(0,1)\\
H_I(t,0,y) &= H_S(t,0,y), & t>0,\ y\in(0,1)\\
\\
p_J\, d_J^- \dfrac{\partial J_I}{\partial x}(t,0,y) &= (1-p_J)\, d_J^+ \dfrac{\partial J_S}{\partial x}(t,0,y), & t>0,\ y\in(0,1)\\
p_A\, d_A^- \dfrac{\partial A_I}{\partial x}(t,0,y) &= (1-p_A)\, d_A^+ \dfrac{\partial A_S}{\partial x}(t,0,y), & t>0,\ y\in(0,1)\\
p_H\, d_H^- \dfrac{\partial H_I}{\partial x}(t,0,y) &= (1-p_H)\, d_H^+ \dfrac{\partial H_S}{\partial x}(t,0,y), & t>0,\ y\in(0,1).
\end{cases}
\tag{6}
$$

The boundary conditions (5) reflect the fact that the entire population evolves inside the domain Ω and that the outside is a hostile habitat. The first three equations in (6) express the continuity of the population density of individuals at Γ, while the last three equations express the skew Brownian motion represented by the probabilities p_J, p_A and p_H which can be different from $1/2$. This implies, in particular, that the flows through the interface Γ are not continuous.

We set

$$\mu_I = p_J\, d_J^-, \quad \alpha_I = p_A\, d_A^- \quad \text{and} \quad \beta_I = p_H\, d_H^-,$$

and

$$\mu_S = (1-p_J)\, d_J^+, \quad \alpha_S = (1-p_A)\, d_A^+ \quad \text{and} \quad \beta_S = (1-p_H)\, d_H^+.$$

Note that μ_I, α_I, β_I and μ_S, α_S, β_S are strictly positive.

We will write system (3) in the form of a perturbed abstract Cauchy problem, see (7) below. To this end, we will adopt the following abstract matrix notations

$$
\begin{aligned}
V(t)(.) := V(t,.) &= \begin{pmatrix} J(t)\\ A(t)\\ H(t)\end{pmatrix}(.) := \begin{pmatrix} J(t,.)\\ A(t,.)\\ H(t,.)\end{pmatrix}\\
&:= \begin{cases}
V_I(t,.) = \begin{pmatrix} J_I(t,.)\\ A_I(t,.)\\ H_I(t,.)\end{pmatrix} & \text{on } \Omega_I\\
\\
V_S(t,.) = \begin{pmatrix} J_S(t,.)\\ A_S(t,.)\\ H_S(t,.)\end{pmatrix} & \text{on } \Omega_S.
\end{cases}
\end{aligned}
$$

Thus, system (3) is written on Ω_I as

$$
\begin{aligned}
&V_I'(t) =\\
&\begin{pmatrix}
d_J^-\Delta + [(1-\tau^-)\sigma_J^- - 1]I & 0 & 0\\
0 & d_A^-\Delta + (\sigma_A^- - 1)I & 0\\
0 & 0 & d_H^-\Delta + [\sigma_H^-(1+\nu f_H^-) - 1]I
\end{pmatrix}
\begin{pmatrix} J_I(t)\\ A_I(t)\\ H_I(t)\end{pmatrix}\\
&+\begin{pmatrix}
\sigma_A^-\, f_A^-\, A_I + \Lambda_J\left(\left(1-\tau^+\right)\, \sigma_J^+\, d_J^+\, \dfrac{\partial J_S(t)}{\partial x}\,|_\Gamma + f_A^+\, \sigma_A^+\, d_A^+\, \dfrac{\partial A_S(t)}{\partial x}\,|_\Gamma + f_A^-\, \sigma_A^-\, d_A^-\, \dfrac{\partial A_I(t)}{\partial x}\,|_\Gamma\right)\\
\tau^-\, \sigma_J^-\, J_I(t) + \Lambda_A\left(\tau^+\, \sigma_J^+\, d_J^+\, \dfrac{\partial J_S(t)}{\partial x}\,|_\Gamma + \sigma_A^+\, d_A^+\, \dfrac{\partial A_S(t)}{\partial x}\,|_\Gamma\right)\\
\Lambda_H\left(\sigma_H^+\, d_H^+\, \dfrac{\partial H_S(t)}{\partial x}\,|_\Gamma + \sigma_H^+\, f_H^+\, d_H^+\, \dfrac{\partial H_S(t)}{\partial x}\,|_\Gamma + (1-\nu)\, f_H^-\, \sigma_H^-\, d_H^-\, \dfrac{\partial H_I(t)}{\partial x}\,|_\Gamma\right)
\end{pmatrix},
\end{aligned}
$$

and on Ω_S as

$$V_S'(t) =$$

$$\begin{pmatrix} d_J^+\Delta + [(1-\tau^+)\sigma_J^+ - 1]I & 0 & 0 \\ 0 & d_A^+\Delta + (\sigma_A^+ - 1)I & 0 \\ 0 & 0 & d_H^+\Delta + [\sigma_H^+(1+f_H^+) - 1]I \end{pmatrix} \begin{pmatrix} J_S(t) \\ A_S(t) \\ H_S(t) \end{pmatrix}$$

$$+ \begin{pmatrix} f_A^+\,\sigma_A^+\,A_S(t) + f_A^-\,\sigma_A^-\,d_A^- \dfrac{\partial A_I(t)}{\partial x}|_\Gamma \\ \tau^+\,\sigma_J^+\,J_S(t) \\ (1-\nu)\,f_H^-\,\sigma_H^-\,d_H^- \dfrac{\partial H_I(t)}{\partial x}|_\Gamma \end{pmatrix}.$$

Then, equations (3), (4), (5) and (6) are written in the following form

$$\begin{cases} V'(t) &= \mathcal{L}V(t) + \mathcal{B}V(t) \\ V(0) &= V_0, \end{cases} \tag{7}$$

where operator $\mathcal{L}$ will act only with respect to the spatial variables x and y and is defined by

$$\begin{cases} D(\mathcal{L}) = \left\{ \begin{array}{l} \varphi = \begin{pmatrix} j \\ a \\ h \end{pmatrix} \in (L^p(\Omega))^3 : \varphi_{|\Omega_I} = \begin{pmatrix} j_I \\ a_I \\ h_I \end{pmatrix} \in \left(W^{2,p}(\Omega_I)\right)^3,\ \varphi_{|\Omega_S} = \begin{pmatrix} j_S \\ a_S \\ h_S \end{pmatrix} \in \left(W^{2,p}(\Omega_S)\right)^3, \\ \begin{pmatrix} j_I \\ a_I \\ h_I \end{pmatrix}(-\ell, y) = \begin{pmatrix} j_I \\ a_I \\ h_I \end{pmatrix}(x,0) = \begin{pmatrix} j_I \\ a_I \\ h_I \end{pmatrix}(x,1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ x \in [-\ell, 0],\ y \in [0,1], \\ \begin{pmatrix} j_S \\ a_S \\ h_S \end{pmatrix}(L, y) = \begin{pmatrix} j_S \\ a_S \\ h_S \end{pmatrix}(x,0) = \begin{pmatrix} j_S \\ a_S \\ h_S \end{pmatrix}(x,1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ x \in [0, L],\ y \in [0,1], \\ \begin{pmatrix} j_I \\ a_I \\ h_I \end{pmatrix}(0,y) = \begin{pmatrix} j_S \\ a_S \\ h_S \end{pmatrix}(0,y) \text{ and } \begin{pmatrix} \mu_I \dfrac{\partial j_I}{\partial x} \\ \alpha_I \dfrac{\partial a_I}{\partial x} \\ \beta_I \dfrac{\partial h_I}{\partial x} \end{pmatrix}(0,y) = \begin{pmatrix} \mu_S \dfrac{\partial j_S}{\partial x} \\ \alpha_S \dfrac{\partial a_S}{\partial x} \\ \beta_S \dfrac{\partial h_S}{\partial x} \end{pmatrix}(0,y),\ y \in [0,1] \end{array} \right\} \\ \mathcal{L}\varphi = \begin{cases} \mathcal{L}_I\varphi & \text{on } \Omega_I \\ \mathcal{L}_S\varphi & \text{on } \Omega_S, \end{cases} \end{cases}$$

with

$$\mathcal{L}_I := \begin{pmatrix} d_J^-\Delta + [(1-\tau^-)\sigma_J^- - 1]I & 0 & 0 \\ 0 & d_A^-\Delta + (\sigma_A^- - 1)I & 0 \\ 0 & 0 & d_H^-\Delta + [\sigma_H^-(1+\nu f_H^-) - 1]I \end{pmatrix},$$

and

$$\mathcal{L}_S := \begin{pmatrix} d_J^+\Delta + [(1-\tau^+)\sigma_J^+ - 1]I & 0 & 0 \\ 0 & d_A^+\Delta + (\sigma_A^+ - 1)I & 0 \\ 0 & 0 & d_H^+\Delta + [\sigma_H^+(1+f_H^+) - 1]I \end{pmatrix}.$$

We have noted all the functions depending on x and y in lower case.

Remark 3.1. The domain of $\mathcal{L}$ takes into account, on the one hand, the dispersal process of the *T. Cruzi* parasite in the domains Ω_I, Ω_S and on the other hand, the boundary and transmission conditions. Its action is defined by those of $\mathcal{L}_I$ and $\mathcal{L}_S$.

Likewise, we set

$$\begin{cases} D(\mathcal{B}) &= D(\mathcal{L}) \\ \mathcal{B}\varphi &= \begin{cases} \mathcal{B}_I\varphi & \text{on } \Omega_I \\ \mathcal{B}_S\varphi & \text{on } \Omega_S, \end{cases} \end{cases}$$

where, for $x \in [-\ell, 0]$ and $y \in [0,1]$, $\mathcal{B}_I$ is defined by

$$(\mathcal{B}_I\varphi)(x,y) =$$
$$\begin{pmatrix} \sigma_A^- f_A^- a_I(x,y) + \Lambda_J \left(\left(1-\tau^+\right) \sigma_J^+ d_J^+ \frac{\partial j_S}{\partial x}(0,y) + f_A^+ \sigma_A^+ d_A^+ \frac{\partial a_S}{\partial x}(0,y) + f_A^- \sigma_A^- d_A^- \frac{\partial a_I}{\partial x}(0,y) \right) \\ \tau^- \sigma_J^- j_I(x,y) + \Lambda_A \left(\tau^+ \sigma_J^+ d_J^+ \frac{\partial j_S}{\partial x}(0,y) + \sigma_A^+ d_A^+ \frac{\partial a_S}{\partial x}(0,y) \right) \\ \Lambda_H \left(\sigma_H^+ d_H^+ \frac{\partial h_S}{\partial x}(0,y) + \sigma_H^+ f_H^+ d_H^+ \frac{\partial h_S}{\partial x}(0,y) + (1-\nu)\, f_H^- \sigma_H^- d_H^- \frac{\partial h_I}{\partial x}(0,y) \right) \end{pmatrix},$$

and for $x \in [0, L]$ and $y \in [0,1]$, $\mathcal{B}_S$ is defined by

$$(\mathcal{B}_S\varphi)(x,y) = \begin{pmatrix} f_A^+ \sigma_A^+ a_S(x,y) + f_A^- \sigma_A^- d_A^- \frac{\partial a_I}{\partial x}(0,y) \\ \tau^+ \sigma_J^+ j_S(x,y) \\ (1-\nu) f_H^- \sigma_H^- d_H^- \frac{\partial h_I}{\partial x}(0,y) \end{pmatrix}.$$

Moreover, due to the transmission conditions, defined in $\mathcal{D}(\mathcal{L})$, we obtain

$$(\mathcal{B}_I\varphi)(x,y) =$$
$$\begin{pmatrix} \sigma_A^- f_A^- a_I(x,y) + \Lambda_J \left(\frac{(1-\tau^+)\, \sigma_J^+ d_J^+ \mu_I}{\mu_S} \frac{\partial j_I}{\partial x}(0,y) + \left(f_A^- \sigma_A^- d_A^- + f_A^+ \sigma_A^+ d_A^+ \frac{\alpha_I}{\alpha_S} \right) \frac{\partial a_I}{\partial x}(0,y) \right) \\ \tau^- \sigma_J^- j_I(x,y) + \Lambda_A \left(\frac{\tau^+ \sigma_J^+ d_J^+ \mu_I}{\mu_S} \frac{\partial j_I}{\partial x}(0,y) + \frac{\sigma_A^+ d_A^+ \alpha_I}{\alpha_S} \frac{\partial a_I}{\partial x}(0,y) \right) \\ \Lambda_H \left(\sigma_H^+ d_H^+ \frac{\left(1+f_H^+\right) \beta_I}{\beta_S} + (1-\nu)\, \sigma_H^- f_H^- d_H^- \right) \frac{\partial h_I}{\partial x}(0,y) \end{pmatrix},$$

and

$$(\mathcal{B}_S\varphi)(x,y) = \begin{pmatrix} f_A^+ \sigma_A^+ a_S(x,y) + \frac{f_A^- \sigma_A^- d_A^- \alpha_S}{\alpha_I} \frac{\partial a_S}{\partial x}(0,y) \\ \tau^+ \sigma_J^+ j_S(x,y) \\ (1-\nu) \frac{f_H^- \sigma_H^- d_H^- \beta_S}{\beta_I} \frac{\partial h_S}{\partial x}(0,y) \end{pmatrix}.$$

Remark 3.2. It is necessary to take $D(\mathcal{B}) = D(\mathcal{L})$ in order to give a sense to the traces of all the partial derivatives with respect to x on Γ. Indeed, for example, for $j_I \in W^{2,p}(\Omega_I)$, we have $\frac{\partial j_I}{\partial x} \in W^{1,p}(\Omega_I)$ and then $\frac{\partial j_I}{\partial x}|_\Gamma \in W^{1-\frac{1}{p},p}(\Gamma)$ (see [17], Theorem 5.10, p. 335), where

$$W^{1-\frac{1}{p},p}(\Gamma) := \left\{ f \in L^p(\Gamma) : (s_1,s_2) \longmapsto \frac{f(s_1)-f(s_2)}{|s_1-s_2|} \in L^p(\Gamma\times\Gamma) \right\},$$

see [33], equation (8), p. 323 and [17], p. 333. The same is true for the other partial derivatives with respect to x which appear in the action of $\mathcal{B}$.

In this paper, we will focus ourselves on the study of the spectral properties of operator $\mathcal{L}$ in order to prove the generation of analytic semigroup. To this end, throughout this work we assume that there exists $r_0 > 0$ such that

$$\max\left(\frac{\sigma_H^-(1+\nu f_H^-)-1}{d_H^-}, \frac{\sigma_H^+(1+f_H^+)-1}{d_H^+}\right) \leqslant r_0 < \pi^2. \tag{8}$$

Remark 3.3. Condition (8) is a general condition for the operator $\mathcal{L}$ to generate an analytic semigroup.
Under the more realistic hypothesis that the disease does not affect the demographic parameters of the vectors and hosts (see [25]), condition (8) is written as:

$$\frac{\sigma(1+f)-1}{d} \leq r_0.$$

where $\sigma = \sigma^+ = \sigma^-$, $f = f^+ = f^-$ and $d = d^+ = d^-$. This inequality is not satisfied for the host population with a high survival probability and important fertility rate in parallel with low mobility. This is the case for a population that exhibits high density. The model is therefore applicable to the case of a host population that exhibits density dependence, which is generally the case in field situations.

Remark 3.4.

1. Note that if $\nu = 0$, assumption (8) becomes

$$\frac{\sigma_H^+(1+f_H^+)-1}{d_H^+} \leqslant r_0 < \pi^2.$$

2. When $\nu > 0$, assumption (8) is obviously satisfied if

$$\sigma_H^- \nu f_H^- \leqslant 1 - \sigma_H^- \quad \text{and} \quad \sigma_H^+ f_H^+ \leqslant 1 - \sigma_H^+.$$

3. Note that this assumption (8) only takes into account the parameters governing the density of hosts.

Our main result in this paper is the following.

Theorem 3.5. Assume that (8) holds. Then, the main dispersal operator $\mathcal{L}$ is linear, closed, with dense domain and generates an analytic semigroup in $\mathcal{E} = [L^p(\Omega)]^3$, where $p \in (1, +\infty)$.

4 The spectral equation

The spectral study of $\mathcal{L}$ is based on the study of the following equation:

$$\mathcal{L}\varphi - \lambda\varphi = \psi \in \mathcal{E} := (L^p(\Omega))^3, \tag{9}$$

where $\varphi \in D(\mathcal{L})$ and λ is a complex number belonging to a sector which will be specified later.

The space $\mathcal{E}$ is normed by

$$\left\| \begin{pmatrix} j \\ a \\ h \end{pmatrix} \right\|_{\mathcal{E}} = \max\left(\|j\|_{L^p(\Omega)}, \|a\|_{L^p(\Omega)}, \|h\|_{L^p(\Omega)} \right),$$

which is equivalent to

$$\max\left(\|j_I\|_{L^p(\Omega_I)} + \|j_S\|_{L^p(\Omega_S)}, \|a_I\|_{L^p(\Omega_I)} + \|a_S\|_{L^p(\Omega_S)}, \|h_I\|_{L^p(\Omega_I)} + \|h_S\|_{L^p(\Omega_S)} \right).$$

Equation (9) writes

$$\mathcal{L}\begin{pmatrix} j \\ a \\ h \end{pmatrix} - \lambda \begin{pmatrix} j \\ a \\ h \end{pmatrix} = \begin{pmatrix} k \\ m \\ n \end{pmatrix},$$

which gives the explicit following system

$$\begin{cases} d_J^- \, \Delta j_I - \Big(\lambda + 1 - (1-\tau^-)\sigma_J^-\Big) \, j_I & = & k_I & \\ d_A^- \, \Delta a_I - \Big(\lambda + 1 - \sigma_A^-\Big) \, a_I & = & m_I & \text{on } \Omega_I \\ d_H^- \, \Delta h_I - \Big(\lambda + 1 - \sigma_H^-(1+\nu f_H^-)\Big) \, h_I & = & n_I, & \\ & & & \\ d_J^+ \, \Delta j_S - \Big(\lambda + 1 - (1-\tau^+)\sigma_J^+\Big) \, j_S & = & k_S & \\ d_A^+ \, \Delta a_S - \Big(\lambda + 1 - \sigma_A^+\Big) \, a_S & = & m_S & \text{on } \Omega_S \\ d_H^+ \, \Delta h_S - \Big(\lambda + 1 - \sigma_H^+(1+f_H^+)\Big) \, h_S & = & n_s, & \end{cases}$$

with

$$\begin{cases} j_I(x,0) = j_I(x,1) = 0 & \\ a_I(x,0) = a_I(x,1) = 0 & x \in [-\ell, 0], \\ h_I(x,0) = h_I(x,1) = 0 & \\ & \\ j_S(x,0) = j_S(x,1) = 0 & \\ a_S(x,0) = a_S(x,1) = 0 & x \in [0, L], \\ h_S(x,0) = h_S(x,1) = 0 & \\ & \\ j_I(-\ell,y) = a_I(-\ell,y) = h_I(-\ell,y) = 0 & \multirow{2}{*}{$y \in [0,1],$} \\ j_S(L,y) = a_S(L,y) = h_S(L,y) = 0, & \end{cases}$$

and

$$\begin{cases} j_I(0,y) = j_S(0,y) & \\ a_I(0,y) = a_S(0,y) & y \in [0,1] \\ h_I(0,y) = h_S(0,y) & \\ & \\ \mu_I \dfrac{\partial j_I}{\partial x}(0,y) = \mu_S \dfrac{\partial j_S}{\partial x}(0,y) & \\ \alpha_I \dfrac{\partial a_I}{\partial x}(0,y) = \alpha_S \dfrac{\partial a_S}{\partial x}(0,y) & y \in [0,1] \\ \beta_I \dfrac{\partial h_I}{\partial x}(0,y) = \beta_S \dfrac{\partial h_S}{\partial x}(0,y). & \end{cases}$$

We then obtain the following three subsystems

$$\begin{cases} \Delta j_I - \left(\dfrac{\lambda}{d_J^-} + \dfrac{1-(1-\tau^-)\,\sigma_J^-}{d_J^-}\right) j_I = \dfrac{k_I}{d_J^-} & \text{on } \Omega_I \\ \Delta j_S - \left(\dfrac{\lambda}{d_J^+} + \dfrac{1-(1-\tau^+)\,\sigma_J^+}{d_J^+}\right) j_S = \dfrac{k_S}{d_J^+} & \text{on } \Omega_S \\ j_I(x,0) = j_I(x,1) = 0 & x \in [-\ell, 0] \\ j_S(x,0) = j_S(x,1) = 0 & x \in [0, L] \\ j_I(-\ell,y) = j_S(L,y) = 0 & y \in [0,1] \\ j_I(0,y) = j_S(0,y) & y \in [0,1] \\ \mu_I \dfrac{\partial j_I}{\partial x}(0,y) = \mu_S \dfrac{\partial j_S}{\partial x}(0,y) & y \in [0,1], \end{cases} \tag{10}$$

$$\left\{\begin{array}{llllll} \Delta a_I - \left(\dfrac{\lambda}{d_A^-} + \dfrac{1-\sigma_A^-}{d_A^-}\right) a_I & = & \dfrac{m_I}{d_A^-} & \text{on } \Omega_I & & \\ \Delta a_S - \left(\dfrac{\lambda}{d_A^+} + \dfrac{1-\sigma_A^+}{d_A^+}\right) a_S & = & \dfrac{m_S}{d_A^+} & \text{on } \Omega_S & & \\ a_I(x,0) & = & a_I(x,1) & = & 0 & x \in [-\ell, 0] \\ a_S(x,0) & = & a_S(x,1) & = & 0 & x \in [0, L] \\ a_I(-\ell,y) & = & a_S(L,y) & = & 0 & y \in [0,1] \\ a_I(0,y) & = & a_S(0,y) & & & y \in [0,1] \\ \alpha_I \dfrac{\partial a_I}{\partial x}(0,y) & = & \alpha_S \dfrac{\partial a_S}{\partial x}(0,y) & & & y \in [0,1]. \end{array}\right. \tag{11}$$

and

$$\left\{\begin{array}{llllll} \Delta h_I - \left(\dfrac{\lambda}{d_H^-} + \dfrac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-}\right) h_I & = & \dfrac{n_I}{d_H^-} & \text{on } \Omega_I & & \\ \Delta h_S - \left(\dfrac{\lambda}{d_H^+} + \dfrac{1-\sigma_H^+(1+f_H^+)}{d_H^+}\right) h_S & = & \dfrac{n_S}{d_H^+} & \text{on } \Omega_S & & \\ h_I(x,0) & = & h_I(x,1) & = & 0 & x \in [-\ell, 0] \\ h_S(x,0) & = & h_S(x,1) & = & 0 & x \in [0, L] \\ h_I(-\ell,y) & = & h_S(L,y) & = & 0 & y \in [0,1] \\ h_I(0,y) & = & h_S(0,y) & & & y \in [0,1] \\ \beta_I \dfrac{\partial h_I}{\partial x}(0,y) & = & \beta_S \dfrac{\partial h_S}{\partial x}(0,y) & & & y \in [0,1]. \end{array}\right. \tag{12}$$

The study of the three systems above is carried out in an analogous manner, with the difference that the study of the last requires hypothesis (8) since, for this system, the following coefficients $1-\sigma_H^-(1+\nu f_H^-)$ and $1-\sigma_H^+(1+f_H^+)$ are not necessarily positive. This is why we focus ourselves on the system governed by the hosts, that is (12).

We will need the prerequisites mentioned in the section below.

5 Prerequisites

5.1 Some properties on complex numbers

In the sequel, we will need the following definition.

Definition 5.1. Let $\omega \in [0, \pi]$, we denote

$$S_\omega = \left\{\begin{array}{ll} \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \omega\} & \text{if } \omega \in (0,\pi], \\ \mathbb{R}_+ \setminus \{0\} & \text{if } \omega = 0. \end{array}\right.$$

Proposition 5.2. Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Then

$$|z_1 + z_2| \geqslant (|z_1| + |z_2|) \left|\cos\left(\frac{\arg(z_1) - \arg(z_2)}{2}\right)\right|.$$

This result is proved in [12], Proposition 4.9, p. 1879.

Proposition 5.3. Let $0 < \alpha < \pi/2$ and $z \in S_\alpha$. We have

1. $|\arg(1-e^{-z}) - \arg(1+e^{-z})| < \alpha$,
2. $|1+e^{-z}| \geqslant 1 - e^{-\pi/(2\tan(\alpha))}$,
3. $\dfrac{|z|\cos(\alpha)}{1+|z|\cos(\alpha)} \leqslant |1-e^{-z}| \leqslant \dfrac{2|z|}{1+|z|\cos(\alpha)}$.

This result is proved in [12], Proposition 4.10, p. 1880.

5.2 Some properties on sectorial operators

This subsection uses the definitions and results of [18].

Definition 5.4. Let $\omega \in (0, \pi)$. A linear operator A on a complex Banach space E is said sectorial with angle ω if

1. $\sigma(A) \subset \overline{S_\omega}$,
2. $M(A, \omega') := \sup\limits_{\lambda \in \mathbb{C} \setminus \overline{S_{\omega'}}} \left\| \lambda (A - \lambda I)^{-1} \right\|_{\mathcal{L}(E)} < +\infty$ for every $\omega' \in (\omega, \pi)$.

Here, $\sigma(A)$ denotes the spectrum of A. When A verifies the two points above, we write $A \in Sect(\omega)$.

Proposition 5.5.

1. If $(-\infty, 0) \subset \rho(A)$ and
$$M(A) := M(A, \pi) := \sup_{t>0} \left\| t(A + tI)^{-1} \right\| < +\infty,$$
then $M(A) \geqslant 1$ and
$$A \in Sect\left(\pi - \arcsin\left(\frac{1}{M(A)}\right)\right).$$
2. If $A \in Sect(\omega_A)$ and $\nu \in (0, 1/2]$, then A^ν is well defined and $A^\nu \in Sect(\nu \omega_A)$. We deduce that $-A^\nu$ generates an analytic semigroup.

5.3 A fundamental property on the H^∞-calculus

This property is based on the results of [7].

We set
$$H^\infty(S_\omega) = \{ f : f \text{ is a bounded holomorphic function on } S_\omega \},$$
with $\omega \in (0, \pi)$. The property is stated as follows

Proposition 5.6. Let A an injective sectorial operator with dense range. If $f \in H^\infty(S_\omega)$ is such that $1/f \in H^\infty(S_\omega)$ and
$$(1/f)(A) \in \mathcal{L}(E),$$
then $f(A)$ is invertible with bounded inverse and
$$[f(A)]^{-1} = (1/f)(A), \tag{13}$$
where $f(A)$ is defined by a Dunford's integral. For more details, see [7] or [12].

5.4 Interpolation spaces

We recall some properties on real interpolation spaces.

Definition 5.7. Let $T_1 : D(T_1) \subset E \longrightarrow E$ a linear operator such that
$$(0, +\infty) \subset \rho(T_1) \quad \text{and} \quad \exists\, C > 0 : \forall\, t > 0, \quad \|t(T_1 - tI)^{-1}\|_{\mathcal{L}(E)} \leqslant C. \tag{14}$$
Let $m \in \mathbb{N} \setminus \{0\}$, $\theta \in (0, 1)$ and $q \in [1, +\infty]$. We will use the following interpolation spaces characterized in [17]:
$$(D(T_1^m), E)_{\theta,q} = (E, D(T_1^m))_{1-\theta,q}.$$
In particular, for $m = 1$, we have the characterisation
$$(D(T_1), E)_{\theta,q} := \left\{ \psi \in E : t \longmapsto t^{1-\theta} \|T_1(T_1 - tI)^{-1}\psi\|_E \in L^q_*(0, +\infty; \mathbb{C}) \right\},$$

where $L_*^q(0,+\infty;\mathbb{C})$ is given by

$$L_*^q(0,+\infty;\mathbb{C}) := \left\{ f \in L^q(0,+\infty) : \left(\int_0^{+\infty} |f(t)|^q \frac{dt}{t} \right)^{1/q} < +\infty \right\}, \quad \text{for } q \in [1,+\infty[,$$

and for $q = +\infty$, by

$$L_*^\infty(0,+\infty;\mathbb{C}) := \left\{ f \text{ is measurable on } (0,+\infty) : \operatorname*{ess\,sup}_{t\in(0,+\infty)} |f(t)| < +\infty \right\}.$$

We also define, for every $m \in \mathbb{N}\setminus\{0\}$

$$(D(T_1),E)_{m+\theta,q} := \{\psi \in D(T_1^m) : T_1^m \psi \in (D(T_1),E)_{\theta,q}\},$$

and

$$(E,D(T_1))_{m+\theta,q} := \{\psi \in D(T_1^m) : T_1^m \psi \in (E,D(T_1))_{\theta,q}\}.$$

Remark 5.8. We remark that T_1^m is closed for every $m \in \mathbb{N}\setminus\{0\}$ as $\rho(T_1) \neq \emptyset$; on the other hand, when $m\theta < 1$, we have

$$(D(T_1^m),E)_{\theta,q} = (E,D(T_1^m))_{1-\theta,q} = (E,D(T_1))_{m-m\theta,q} = (D(T_1),E)_{(m-1)+m\theta,q} \subset D(T_1^{m-1});$$

see details in [24], (2.1.13), p. 43, or [17], p. 676, Theorem 6.

Remark 5.9. Let $a,b \in \mathbb{R}$, $a<b$, $n \in \mathbb{N}\setminus\{0\}$ and T_2 the generator of a bounded analytic semigroup in E; we recall

$$\begin{cases} x \longmapsto e^{(x-a)T_2}\psi \in L^p(a,b;E) \text{ and } x \longmapsto e^{(b-x)T_2}\psi \in L^p(a,b;E) \text{ for every } \psi \in E, \\ x \longmapsto T_2^n e^{(x-a)T_2}\psi \in L^p(a,b;E) \Longleftrightarrow \psi \in (D(T_2^n),E)_{\frac{1}{np},p}, \\ x \longmapsto T_2^n e^{(b-x)T_2}\psi \in L^p(a,b;E) \Longleftrightarrow \psi \in (D(T_2^n),E)_{\frac{1}{np},p}, \end{cases} \tag{15}$$

where $p \in (1,+\infty)$; see Theorem p. 96 in [33] for the two last statements.

Moreover, by the reiteration Theorem, it follows that, the three following properties are equivalent

1. $x \longmapsto e^{(x-a)T_2}\psi \in W^{n,p}(a,b;E) \cap L^p(a,b;D(T_2^n))$,
2. $x \longmapsto e^{(b-x)T_2}\psi \in W^{n,p}(a,b;E) \cap L^p(a,b;D(T_2^n))$,
3. $\psi \in (D(T_2),E)_{n-1+\frac{1}{p},p}$.

5.5 Recall on the Dore-Venni Theorem

We now recall the famous Theorem of Dore and Venni, see [13], where the complex Banach space E is supposed to be UMD, see [4] and [5].

Definition 5.10. Let $\theta \in [0,\pi)$. We denote by BIP(E,θ), the class of injective sectorial operators $-T_3$ such that

i) $\overline{D(T_3)} = \overline{R(T_3)} = E$,

ii) $\forall\ s \in \mathbb{R}, \quad (-T_3)^{is} \in \mathcal{L}(E)$,

iii) $\exists\ C \geq 1,\ \forall\ s \in \mathbb{R}, \quad \|(-T_3)^{is}\|_{\mathcal{L}(E)} \leq Ce^{|s|\theta}$,

see [30], p. 430.

Theorem 5.11. Let $-T_3 \in \text{BIP}(E,\theta)$ with $\theta \in (0,\pi/2)$ and $g \in L^p(a,b;E)$, where $p \in (1,+\infty)$ and $a,b \in \mathbb{R}$ with $a<b$. Then, for almost every $x \in (a,b)$, we have

$$\int_a^x e^{(x-s)T_3} g(s)\,ds \in D(T_3) \quad \text{and} \quad \int_x^b e^{(s-x)T_3} g(s)\,ds \in D(T_3).$$

Moreover

$$x \longmapsto T_3 \int_a^x e^{(x-s)T_3} g(s)\,ds \in L^p(a,b;E) \quad \text{and} \quad x \longmapsto T_3 \int_x^b e^{(s-x)T_3} g(s)\,ds \in L^p(a,b;E).$$

6 Operational formulation of system (11)

We consider, in the UMD Banach space $X = L^p(0,1)$, where $p \in (1,+\infty)$, linear operators Q, Q^- and Q^+ defined by

$$\begin{cases} D(Q) & = & \{\phi \in W^{2,p}(0,1) : \phi(0) = \phi(1) = 0\} \\ (Q\phi)(y) & = & \phi''(y), \end{cases}$$

$$\begin{cases} D(Q^-) & = & \{\phi \in W^{2,p}(0,1) : \phi(0) = \phi(1) = 0\} \\ (Q^-\phi)(y) & = & \phi''(y) - \dfrac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-}\phi(y), \end{cases}$$

and

$$\begin{cases} D(Q^+) & = & \{\phi \in W^{2,p}(0,1) : \phi(0) = \phi(1) = 0\} \\ (Q^+\phi)(y) & = & \phi''(y) - \dfrac{1-\sigma_H^+(1+f_H^+)}{d_H^+}\phi(y). \end{cases}$$

Proposition 6.1. Assume that (8) holds, then operators Q, Q^- and Q^+ are linear closed with dense domains in X and verify

$$\begin{cases} \forall\,\eta \in (0,\pi),\ S_{\pi-\eta} \cup \{0\} \subset \rho(Q) \\ \exists\,C > 0,\ \forall\,z \in S_{\pi-\eta} \cup \{0\}, \quad \|(Q - zI)^{-1}\|_{\mathcal{L}(X)} \leqslant \dfrac{C}{1+|z|}, \end{cases} \tag{16}$$

and

$$\begin{cases} \forall\,\eta \in (0,\pi),\ S_{\pi-\eta} \cup \{0\} \subset \rho(Q^\pm) \\ \exists\,C > 0,\ \forall\,z \in S_{\pi-\eta} \cup \{0\}, \quad \|(Q^\pm - zI)^{-1}\|_{\mathcal{L}(X)} \leqslant \dfrac{C}{1+|z|}. \end{cases} \tag{17}$$

Moreover, there exists an open ball $B(0,\delta)$, $\delta > 0$, such that $\overline{B(0,\delta)} \subset \rho(Q)$, $\overline{B(0,\delta)} \subset \rho(Q^\pm)$ and the estimates (16) and (17) remain true in $S_{\pi-\eta} \cup \overline{B(0,\delta)}$. Here, $\rho(Q)$ and $\rho(Q^\pm)$ denote respectively the resolvent sets of Q and $Q^\pm$.

Proof. The proof of (16) is well known, while the one of (17) follows essentially from (8) and the fact that the second derivative operator with Dirichlet boundary conditions admits for eigenvalues the sequence $-k^2\pi^2$, $k \in \mathbb{N}^*$. □

Remark 6.2. The definition of sectorial operator A has been given in Definition 5.4. We recall that this notion corresponds, in the Hilbert case, to the positivity of the scalar product $< Au, u >$. A typical example of such an operator A is an elliptic operator.

Here, $-Q$, $-Q^-$ and $-Q^+$ are clearly sectorial and satisfy Definition 5.4.

Using the usual notation for vector-valued functions, we set

$$\begin{cases} h_I(x)(y) & := & h_I(x,y) \\ h_S(x)(y) & := & h_S(x,y) \end{cases} \quad \text{and} \quad \begin{cases} g_I(x)(y) & := & g_I(x,y) = \dfrac{n_I}{d_H^-}(x,y) \\ g_S(x)(y) & := & g_S(x,y) = \dfrac{n_S}{d_H^+}(x,y). \end{cases}$$

Thus, system (11) can be written as

$$\begin{cases} h_I''(x) + Q^-h_I(x) - \dfrac{\lambda}{d_H^-}h_I(x) & = & g_I(x), \quad \text{a.e. } x \in (-\ell,0) \\ h_S''(x) + Q^+h_S(x) - \dfrac{\lambda}{d_H^+}h_S(x) & = & g_S(x), \quad \text{a.e. } x \in (0,L) \\ h_I(-\ell) & = & h_S(L) = 0 \\ h_I(0) & = & h_S(0) \\ \beta_I h_I'(0) & = & \beta_S h_S'(0), \end{cases} \tag{18}$$

where

$$g_I \in L^p(-\ell, 0; X) = L^p(-\ell, 0; L^p(0,1)) = L^p(\Omega_I),$$

and

$$g_S \in L^p(0, L; X) = L^p(0, L; L^p(0,1)) = L^p(\Omega_S).$$

7 Spectral properties of operators $Q^\pm$

Now, we have to specify the sector of the spectral parameter λ. In all the sequel

$$\lambda \in S_{\pi-\varepsilon}, \tag{19}$$

where $\varepsilon \in (0, \pi/2)$ is fixed and will be specified later. We set

$$Q_\lambda^- = Q - \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} I - \frac{\lambda}{d_H^-} I = Q^- - \frac{\lambda}{d_H^-} I \quad \text{and} \quad Q_\lambda^+ = Q - \frac{1-\sigma_H^+(1+f_H^+)}{d_H^+} I - \frac{\lambda}{d_H^+} I = Q^+ - \frac{\lambda}{d_H^+} I;$$

thus, we have

$$D(Q_\lambda^-) = D(Q^-) = D(Q^+) = D(Q_\lambda^+).$$

According to Proposition 6.1, operators $-Q^\pm$ are sectorial in X. The same is true for operators $-Q_\lambda^\pm$. Indeed, we have $(-\infty, 0] \subset \rho(-Q_\lambda^\pm)$. We set

$$M(-Q_\lambda^\pm) := M(-Q_\lambda^\pm, \pi) := \sup_{t>0} \|t(-Q_\lambda^\pm + tI)^{-1}\|_{\mathcal{L}(X)}.$$

Due to Proposition 5.5, for all $\lambda \in S_{\pi-\varepsilon}$, we obtain

$$M(-Q_\lambda^\pm) \leqslant \sup_{t>0} \left(\frac{t}{\cos\left(\frac{1}{2}\arg\left(\frac{\lambda}{d_H^\pm} + t\right)\right)} \frac{1}{\left|\frac{\lambda}{d_H^\pm} + t\right|} \right).$$

Two cases are possible:

1. If $|\arg(\lambda)| < \pi/2$, then:
$$\forall\, t > 0, \qquad \left|\frac{\lambda}{d_H^\pm} + t\right| \geqslant t.$$

2. If $\pi/2 \leqslant |\arg(\lambda)| < \pi - \varepsilon$, then
$$\forall\, t > 0, \qquad \left|\frac{\lambda}{d_H^\pm} + t\right| \geqslant t\sin(\alpha) \geqslant t\sin(\varepsilon),$$
since $\alpha \in (\varepsilon, \pi/2]$.

Thus, in these two cases, there exists a constant $C > 0$ independent of λ such that

$$M(-Q_\lambda^\pm) \leqslant \frac{C}{\cos\left(\dfrac{\pi-\varepsilon}{2}\right)} = \frac{C}{\sin(\varepsilon)} < +\infty,$$

so

$$-Q_\lambda^\pm \in Sect\left(\pi - \arcsin\left(\frac{1}{M(-Q_\lambda^\pm)}\right)\right).$$

We deduce that the two following operators

$$P_\lambda^- = -\left(-\left(Q^- - \frac{\lambda}{d_H^-} I\right)\right)^{1/2} \quad \text{and} \quad P_\lambda^+ = -\left(-\left(Q^+ - \frac{\lambda}{d_H^+} I\right)\right)^{1/2},$$

are well defined and have the same domain

$$D(P_\lambda^-) = D(P_\lambda^+) = D\left((-Q^-)^{1/2}\right) = D\left(\left(-Q^+\right)^{1/2}\right).$$

It is well known that operators $-(-Q^-)^{1/2}$ and $-(-Q^+)^{1/2}$ generate analytic semigroups in X, see [3].

Using Lemma 4.2 in [15] as well as estimates (28) and (29) in [15], there exist $\varepsilon^\pm > 0$ and $C^\pm > 0$ independent of λ such that

$$\begin{cases} \forall\, z \in \left\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leqslant \dfrac{\pi}{2} + \varepsilon^\pm \right\} \\ \left\| (P_\lambda^\pm - zI)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \dfrac{C^\pm}{\sqrt{1+|\lambda|+|z|}}. \end{cases} \tag{20}$$

8 Resolution of system (18)

From this section, we will assume in the sequel that

$$\lambda \in S_{2(\pi-\varepsilon)/3},$$

where $\varepsilon \in (0,\pi/2)$ is fixed small enough; this will allow us to apply the H^∞-calculus for the resolution of system (18).

8.1 Calculus of the determinant operator

Using operators P_λ^- and P_λ^+, system (18) writes as follows

$$\begin{cases} h_I''(x) - (P_\lambda^-)^2 h_I(x) &= g_I(x), \quad \text{a.e. } x \in (-\ell, 0) \\ h_S''(x) - (P_\lambda^+)^2 h_S(x) &= g_S(x), \quad \text{a.e. } x \in (0, L) \\ h_I(-\ell) = h_S(L) = 0 \\ h_I(0) = h_S(0) \\ \beta_I h_I'(0) = \beta_S h_S'(0). \end{cases} \tag{21}$$

Applying a similar method as in [16], the solution (h_I, h_S) of system (21) writes as

$$\begin{cases} h_I(x) = e^{(x+\ell)P_\lambda^-}\gamma_I + e^{-xP_\lambda^-}\delta_I + w_I(g_I)(x), & x \in (-\ell, 0) \\ h_S(x) = e^{xP_\lambda^+}\gamma_S + e^{(L-x)P_\lambda^+}\delta_S + w_S(g_S)(x), & x \in (0, L), \end{cases}$$

where

$$\begin{cases} w_I(g_I)(x) = \dfrac{1}{2}\displaystyle\int_{-\ell}^x e^{(x-t)P_\lambda^-}(P_\lambda^-)^{-1} g_I(t)\, dt + \dfrac{1}{2}\displaystyle\int_x^0 e^{(t-x)P_\lambda^-}(P_\lambda^-)^{-1} g_I(t)\, dt, & x \in (-\ell, 0) \\ w_S(g_S)(x) = \dfrac{1}{2}\displaystyle\int_0^x e^{(x-t)P_\lambda^+}(P_\lambda^+)^{-1} g_S(t)\, dt + \dfrac{1}{2}\displaystyle\int_x^L e^{(t-x)P_\lambda^+}(P_\lambda^+)^{-1} g_S(t)\, dt, & x \in (0, L). \end{cases}$$

It is easy to see that all these integrals are well defined due to the semigroups properties. We will prove later that functions h_I and h_S satisfy the following optimal regularity

$$\begin{cases} h_I \in W^{2,p}(-\ell, 0; X) \cap L^p\left(-\ell, 0; D\left((P_\lambda^-)^2\right)\right) \\ h_S \in W^{2,p}(0, L; X) \cap L^p\left(0, L; D\left((P_\lambda^+)^2\right)\right). \end{cases}$$

Now, we have to find the constants γ_I, γ_S, δ_I and δ_S. Thanks to the boundary conditions, we have

$$\begin{cases} 0 = h_I(-\ell) = \gamma_I + e^{\ell P_\lambda^-}\delta_I + w_I(g_I)(-\ell) \\ 0 = h_S(L) = e^{LP_\lambda^+}\gamma_S + \delta_S + w_S(g_S)(L), \end{cases}$$

where

$$\begin{cases} w_I(g_I)(-\ell) &= \dfrac{1}{2}\displaystyle\int_{-\ell}^{0} e^{(t+\ell)P_\lambda^-}(P_\lambda^-)^{-1}g_I(t)\,dt \\ w_S(g_S)(L) &= \dfrac{1}{2}\displaystyle\int_{0}^{L} e^{(L-t)P_\lambda^+}(P_\lambda^+)^{-1}g_S(t)\,dt. \end{cases}$$

Hence

$$\begin{cases} \gamma_I &= -e^{\ell P_\lambda^-}\delta_I - w_I(g_I)(-\ell) \\ \delta_S &= -e^{LP_\lambda^+}\gamma_S - w_S(g_S)(L). \end{cases} \tag{22}$$

On the other hand, we have

$$\begin{cases} h_I'(x) &= P_\lambda^- e^{(x+\ell)P_\lambda^-}\gamma_I - P_\lambda^- e^{-xP_\lambda^-}\delta_I + w_I'(g_I)(x), \quad x \in (-\ell,0) \\ h_S'(x) &= P_\lambda^+ e^{xP_\lambda^+}\gamma_S - P_\lambda^+ e^{(L-x)P_\lambda^+}\delta_S + w_S'(g_S)(x), \quad x \in (0,L), \end{cases}$$

with

$$\begin{cases} w_I'(g_I)(x) &= \dfrac{1}{2}\displaystyle\int_{-\ell}^{x} e^{(x-t)P_\lambda^-}g_I(t)\,dt - \dfrac{1}{2}\displaystyle\int_{x}^{0} e^{(t-x)P_\lambda^-}g_I(t)\,dt, \quad x \in (-\ell,0) \\ w_S'(g_S)(x) &= \dfrac{1}{2}\displaystyle\int_{0}^{x} e^{(x-t)P_\lambda^+}g_S(t)\,dt - \dfrac{1}{2}\displaystyle\int_{x}^{L} e^{(t-x)P_\lambda^+}g_S(t)\,dt, \quad x \in (0,L). \end{cases}$$

From the transmission conditions, we formally deduce that

$$\begin{cases} e^{\ell P_\lambda^-}\gamma_I + \delta_I + w_I(g_I)(0) &= \gamma_S + e^{LP_\lambda^+}\delta_S + w_S(g_S)(0) \\ \beta_I\left(P_\lambda^- e^{\ell P_\lambda^-}\gamma_I - P_\lambda^-\delta_I + w_I'(g_I)(0)\right) &= \beta_S\left(P_\lambda^+\gamma_S - P_\lambda^+ e^{LP_\lambda^+}\delta_S + w_S'(g_S)(0)\right), \end{cases} \tag{23}$$

where

$$\begin{cases} w_I(g_I)(0) &= \dfrac{1}{2}\displaystyle\int_{-\ell}^{0} e^{-tP_\lambda^-}(P_\lambda^-)^{-1}g_I(t)\,dt \\ w_S(g_S)(0) &= \dfrac{1}{2}\displaystyle\int_{0}^{L} e^{tP_\lambda^+}(P_\lambda^+)^{-1}g_S(t)\,dt, \end{cases}$$

and

$$\begin{cases} w_I'(g_I)(0) &= \dfrac{1}{2}\displaystyle\int_{-\ell}^{0} e^{-tP_\lambda^-}g_I(t)\,dt &= P_\lambda^- w_I(g_I)(0) \\ w_S'(g_S)(0) &= -\dfrac{1}{2}\displaystyle\int_{0}^{L} e^{tP_\lambda^+}g_S(t)\,dt &= -P_\lambda^+ w_S(g_S)(0). \end{cases} \tag{24}$$

We will see, later, that the equalities in (23) are well defined since $\delta_I \in D(P_\lambda^-)$ and $\gamma_S \in D(P_\lambda^+)$. On the other hand, we know that integrals in (24) belong respectively to $D(P_\lambda^-)$ and $D(P_\lambda^+)$ from Proposition 1.2, (ii), p. 20 in [31].

Applying $(P_\lambda^-)^{-1}$ on the second line of system (23), we obtain

$$\begin{cases} e^{\ell P_\lambda^-}\gamma_I + \delta_I + w_I(g_I)(0) &= \gamma_S + e^{LP_\lambda^+}\delta_S + w_S(g_S)(0) \\ \beta_I\left(e^{\ell P_\lambda^-}\gamma_I - \delta_I + (P_\lambda^-)^{-1}w_I'(g_I)(0)\right) &= \beta_S\left((P_\lambda^-)^{-1}P_\lambda^+\gamma_S - (P_\lambda^-)^{-1}P_\lambda^+ e^{LP_\lambda^+}\delta_S\right) \\ &\quad +\beta_S(P_\lambda^-)^{-1}w_S'(g_S)(0), \end{cases}$$

and, since $(P_\lambda^-)^{-1}$ and P_λ^+ commute on the domain of P_λ^+, we have

$$\begin{cases} e^{\ell P_\lambda^-}\gamma_I + \delta_I + w_I(g_I)(0) &= \gamma_S + e^{LP_\lambda^+}\delta_S + w_S(g_S)(0) \\ \beta_I\left(e^{\ell P_\lambda^-}\gamma_I - \delta_I + (P_\lambda^-)^{-1}w_I'(g_I)(0)\right) &= \beta_S\left(P_\lambda^+(P_\lambda^-)^{-1}\gamma_S - P_\lambda^+(P_\lambda^-)^{-1}e^{LP_\lambda^+}\delta_S\right) \\ &\quad +\beta_S(P_\lambda^-)^{-1}w_S'(g_S)(0). \end{cases}$$

Using (24), we deduce that

$$\left\{\begin{array}{lcl} e^{\ell P_\lambda^-}\gamma_I + \delta_I + w_I(g_I)(0) & = & \gamma_S + e^{LP_\lambda^+}\delta_S + w_S(g_S)(0) \\ \beta_I\left(e^{\ell P_\lambda^-}\gamma_I - \delta_I + w_I(g_I)(0)\right) & = & \beta_S\, P_\lambda^+(P_\lambda^-)^{-1}\left(\gamma_S - e^{LP_\lambda^+}\delta_S - w_S(g_S)(0)\right). \end{array}\right. \tag{25}$$

We set

$$R_I = w_I(g_I)(0) - e^{\ell P_\lambda^-} w_I(g_I)(-\ell) = \frac{1}{2}\int_{-\ell}^0 e^{-tP_\lambda^-}\left(I - e^{2(t+\ell)P_\lambda^-}\right)(P_\lambda^-)^{-1} g_I(t)\,dt,$$

and

$$R_S = w_S(g_S)(0) - e^{LP_\lambda^+} w_S(g_S)(L) = \frac{1}{2}\int_0^L e^{tP_\lambda^+}\left(I - e^{2(L-t)P_\lambda^+}\right)(P_\lambda^+)^{-1} g_S(t)\,dt.$$

Using (22), system (25) becomes

$$\left\{\begin{array}{lcl} \left(I - e^{2\ell P_\lambda^-}\right)\delta_I + R_I & = & \left(I - e^{2LP_\lambda^+}\right)\gamma_S + R_S \\ \beta_I\left(-\left(I + e^{2\ell P_\lambda^-}\right)\delta_I + R_I\right) & = & \beta_S\, P_\lambda^+(P_\lambda^-)^{-1}\left(\left(I + e^{2LP_\lambda^+}\right)\gamma_S - R_S\right), \end{array}\right.$$

hence

$$\left\{\begin{array}{lcl} \left(I - e^{2\ell P_\lambda^-}\right)\delta_I - \left(I - e^{2LP_\lambda^+}\right)\gamma_S & = & R_S - R_I \\ \beta_I\left(I + e^{2\ell P_\lambda^-}\right)\delta_I + \beta_S\, P_\lambda^+(P_\lambda^-)^{-1}\left(I + e^{2LP_\lambda^+}\right)\gamma_S & = & \beta_I\, R_I + \beta_S\, P_\lambda^+(P_\lambda^-)^{-1} R_S. \end{array}\right. \tag{26}$$

Note that all the coefficient operators of system (26) are bounded. Therefore, the abstract determinant of this system is given by

$$\begin{array}{rcl} D_\lambda & = & \beta_I\left(I + e^{2\ell P_\lambda^-}\right)\left(I - e^{2LP_\lambda^+}\right) + \beta_S\, P_\lambda^+\left(P_\lambda^-\right)^{-1}\left(I - e^{2\ell P_\lambda^-}\right)\left(I + e^{2LP_\lambda^+}\right) \\ & = & \beta_I\left(I + e^{2\ell P_\lambda^-}\right)\left(I - e^{2LP_\lambda^+}\right)\Pi_\lambda, \end{array}$$

where

$$\Pi_\lambda = I + \frac{\beta_S}{\beta_I}\, P_\lambda^+(P_\lambda^-)^{-1}\left(I - e^{2\ell P_\lambda^-}\right)\left(I + e^{2LP_\lambda^+}\right)\left(I + e^{2\ell P_\lambda^-}\right)^{-1}\left(I - e^{2LP_\lambda^+}\right)^{-1}.$$

We used the fact that operators $\left(I - e^{2LP_\lambda^+}\right)$ and $\left(I + e^{2\ell P_\lambda^-}\right)$ are boundedly invertible respectively in virtue of Proposition 2.3.6, p. 60, in [24] and Lemma 5.2, p. 1883 in [12].

8.2 Inversion of the determinant

We recall that $\lambda \in S_{2(\pi-\varepsilon)/3}$ and

$$P_\lambda^- = -\left(-\left(Q - \frac{1 - \sigma_H^-(1 + \nu f_H^-)}{d_H^-} I - \frac{\lambda}{d_H^-} I\right)\right)^{1/2} \quad \text{and} \quad P_\lambda^+ = -\left(-\left(Q - \frac{1 - \sigma_H^+(1 + f_H^+)}{d_H^+} I - \frac{\lambda}{d_H^+} I\right)\right)^{1/2}.$$

We set

$$\lambda_- = -\frac{1 - \sigma_H^-(1 + \nu f_H^-)}{d_H^-} - \frac{\lambda}{d_H^-} \quad \text{and} \quad \lambda_+ = -\frac{1 - \sigma_H^+(1 + f_H^+)}{d_H^+} - \frac{\lambda}{d_H^+}.$$

Thus

$$P_\lambda^- = -\left(-Q - \lambda_- I\right)^{1/2} \quad \text{and} \quad P_\lambda^+ = -\left(-Q - \lambda_+ I\right)^{1/2}.$$

For all $z \in \pi^2 + S_{(\pi-\varepsilon)/3}$, we define the following function

$$f_\lambda(z) = 1 + \frac{\beta_S}{\beta_I}\frac{\sqrt{z - \lambda_+}}{\sqrt{z - \lambda_-}}\frac{\left(1 - e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1 + e^{-2L\sqrt{z-\lambda_+}}\right)}{\left(1 + e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1 - e^{-2L\sqrt{z-\lambda_+}}\right)}, \tag{27}$$

in order to have

$$f_\lambda(-Q) = \Pi_\lambda.$$

We will analyze the following complex quantities

$$Z_+ = \sqrt{z-\lambda_+} \quad \text{and} \quad Z_- = \sqrt{z-\lambda_-},$$

as well as

$$\left(1 \pm e^{-2L\sqrt{z-\lambda_+}}\right) \quad \text{and} \quad \left(1 \pm e^{-2\ell\sqrt{z-\lambda_-}}\right)$$

for

$$z \in \pi^2 + S_{(\pi-\varepsilon)/3} \quad \text{and} \quad \lambda \in S_{2(\pi-\varepsilon)/3}.$$

We will then need the following essential technical lemma in order to apply Proposition 5.3. Then

$$Z_+ = \sqrt{z-\lambda_+} \in S_{(\pi-\varepsilon)/3} \subset S_{\pi/3} \quad \text{and} \quad Z_- = \sqrt{z-\lambda_-} \in S_{(\pi-\varepsilon)/3} \subset S_{\pi/3}, \tag{28}$$

thus

$$2LZ_+ = 2L\sqrt{z-\lambda_+} \in S_{(\pi-\varepsilon)/3} \subset S_{\pi/3} \quad \text{and} \quad 2\ell Z_- = 2\ell\sqrt{z-\lambda_-} \in S_{(\pi-\varepsilon)/3} \subset S_{\pi/3}.$$

Lemma 8.1. Let $z \in \pi^2 + S_{(\pi-\varepsilon)/3}$ and $\lambda \in S_{2(\pi-\varepsilon)/3}$, with $0 < \varepsilon < \pi/2$ fixed. We have

1. $|Z_\pm| \geqslant \sqrt{\dfrac{\sqrt{3}\,(\pi^2 - r_0)}{2}\sin(\varepsilon/2)}$,
2. $\begin{cases} \left|\arg\left(1-e^{-2LZ_+}\right) - \arg\left(1+e^{-2LZ_+}\right)\right| & < \dfrac{\pi}{3} - \dfrac{\varepsilon}{3} \\ \left|\arg\left(1-e^{-2\ell Z_-}\right) - \arg\left(1+e^{-2\ell Z_-}\right)\right| & < \dfrac{\pi}{3} - \dfrac{\varepsilon}{3}, \end{cases}$
3. $\begin{cases} \left|1+e^{-2LZ_+}\right| & \geqslant 1 - e^{-\pi/(2\tan((\pi-\varepsilon)/3))} \geqslant 1 - e^{-\pi/2\sqrt{3}} \\ \left|1+e^{-2\ell Z_-}\right| & \geqslant 1 - e^{-\pi/(2\tan((\pi-\varepsilon)/3))} \geqslant 1 - e^{-\pi/2\sqrt{3}}, \end{cases}$
4. $\begin{cases} \dfrac{L|Z_+|}{1+L|Z_+|} & \leqslant |1-e^{-2LZ_+}| \leqslant \dfrac{4L|Z_+|}{1+L|Z_+|} \\ \dfrac{\ell|Z_-|}{1+\ell|Z_-|} & \leqslant |1-e^{-2\ell Z_-}| \leqslant \dfrac{4\ell|Z_-|}{1+\ell|Z_-|}. \end{cases}$

Proof. Statements 2., 3., and 4. are proved by a direct application of Proposition 5.3.

Let us show statement 1. Since $z \in \pi^2 + S_{(\pi-\varepsilon)/3}$, there exists $z' \in S_{(\pi-\varepsilon)/3}$ such that

$$z = z' + \pi^2.$$

We set

$$z_- = z + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} \quad \text{and} \quad z_+ = z + \frac{1-\sigma_H^+(1+f_H^+)}{d_H^+},$$

thus

$$z - \lambda_- = z_- + \frac{\lambda}{d_H^-} \quad \text{and} \quad z - \lambda_+ = z_+ + \frac{\lambda}{d_H^+}.$$

If

$$\frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} < 0,$$

then

$$|z_-| = \left|z + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-}\right| = \left|z' + \pi^2 + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-}\right| = \left|z' + \pi^2 - \frac{\sigma_H^-(1+\nu f_H^-)-1}{d_H^-}\right|.$$

From assumption (8), we have

$$\pi^2 - \frac{\sigma_H^-(1+\nu f_H^-)-1}{d_H^-} \geqslant \pi^2 - r_0 > 0,$$

and due to Proposition 5.2, we obtain

$$\begin{aligned}
|z_-| &= \left| z' + \pi^2 - \frac{\sigma_H^-(1+\nu f_H^-)-1}{d_H^-} \right| \\
&\geqslant \left(|z'| + \left| \pi^2 - \frac{\sigma_H^-(1+\nu f_H^-)-1}{d_H^-} \right| \right) \cos\left(\frac{\arg(z')}{2}\right) \\
&\geqslant \left| \pi^2 - \frac{\sigma_H^-(1+\nu f_H^-)-1}{d_H^-} \right| \cos\left(\frac{\arg(z')}{2}\right) \\
&\geqslant (\pi^2 - r_0)\frac{\sqrt{3}}{2}.
\end{aligned}$$

Now, if

$$\frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} > 0,$$

then

$$|z_-| = \left| z + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} \right| = \left| z' + \pi^2 + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} \right|,$$

and from Proposition 5.2, it follows that

$$\begin{aligned}
|z_-| &= \left| z' + \pi^2 + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} \right| \\
&\geqslant \left(|z'| + \left| \pi^2 + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} \right| \right) \cos\left(\frac{\arg(z')}{2}\right) \\
&\geqslant \left(\pi^2 + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} \right) \cos\left(\frac{\arg(z')}{2}\right) \\
&\geqslant \pi^2 \cos\left(\frac{\arg(z')}{2}\right) \\
&\geqslant (\pi^2 - r_0)\cos\left(\frac{\pi}{6}\right) \\
&\geqslant (\pi^2 - r_0)\frac{\sqrt{3}}{2}.
\end{aligned}$$

Note that in this case, we have not used assumption (8).

Similarly, we deduce that

$$|z_+| = \left| z + \frac{1-\sigma_H^+(1+f_H^+)}{d_H^+} \right| \geqslant (\pi^2 - r_0)\frac{\sqrt{3}}{2};$$

thus $z_\pm \in S_{(\pi-\varepsilon)/3}$. We then obtain

$$\begin{aligned}
|z - \lambda_\pm| = \left| z_\pm + \frac{\lambda}{d_H^\pm} \right| &\geqslant \left(|z_\pm| + \left| \frac{\lambda}{d_H^\pm} \right| \right) \left| \cos\left(\frac{\arg(z_\pm) - \arg(\lambda)}{2}\right) \right| \\
&\geqslant |z_\pm| \left| \cos\left(\frac{\arg(z_\pm) - \arg(\lambda)}{2}\right) \right| \\
&\geqslant \frac{\sqrt{3}\,(\pi^2 - r_0)}{2} \left| \cos\left(\frac{\arg(z_\pm) - \arg(\lambda)}{2}\right) \right|.
\end{aligned}$$

On the other hand

$$|\arg(z_\pm)-\arg(\lambda)| \leqslant |\arg(z_\pm)|+|\arg(\lambda)| < \frac{\pi-\varepsilon}{3}+\frac{2(\pi-\varepsilon)}{3}=\pi-\varepsilon.$$

Hence

$$\begin{aligned}|z-\lambda_\pm| &\geqslant \frac{\sqrt{3}\,(\pi^2-r_0)}{2}\left|\cos\left(\frac{\arg(z_\pm)-\arg(\lambda)}{2}\right)\right| \\ &> \frac{\sqrt{3}\,(\pi^2-r_0)}{2}\cos\left(\frac{\pi}{2}-\frac{\varepsilon}{2}\right)=\frac{\sqrt{3}\,(\pi^2-r_0)}{2}\sin\left(\frac{\varepsilon}{2}\right)>0.\end{aligned}$$

□

Let us go back to the function f_λ, given by (27), where λ is fixed in $S_{2(\pi-\varepsilon)/3}$.

Proposition 8.2. Function f_λ, defined by (27), is analytic and bounded on $\pi^2+S_{(\pi-\varepsilon)/3}$.

Proof. We first prove that the denominator in the expression of f_λ never vanishes.

For $z\in\pi^2+S_{(\pi-\varepsilon)/3}$, using Lemma 8.1, we have

$$\left|\sqrt{z-\lambda_-}\left(1+e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1-e^{-2L\sqrt{z-\lambda_+}}\right)\right| \geqslant \frac{\sqrt{3}\,(\pi^2-r_0)L}{2}\,\frac{\left(1-e^{-\pi/2\sqrt{3}}\right)\sin(\varepsilon/2)}{1+L\sqrt{|z-\lambda_+|}}>0.$$

We deduce that function f_λ is holomorphic on $\pi^2+S_{(\pi-\varepsilon)/3}$ and moreover it is bounded since

$$|f_\lambda(z)|\leqslant 1+\frac{C\,\beta_S}{\beta_I\left(1-e^{-\pi/2\sqrt{3}}\right)}\sqrt{\frac{z-\lambda_+}{z-\lambda_-}\,\frac{1+L\sqrt{|z-\lambda_+|}}{L\sqrt{|z-\lambda_+|}}},$$

where $C>0$ is a constant independent of z and λ. □

We conclude that

$$f_\lambda(-Q)=\Pi_\lambda.$$

Proposition 8.3. Let $\varepsilon_0\in\left(0,\dfrac{\pi}{8}\right)$. Then operator Π_λ is boundedly invertible for all $\lambda\in S_{4(\pi-\varepsilon_0)/7}$. It follows that

$$D_\lambda^{-1}=\frac{1}{\beta_I}\left(I+e^{2\ell P_\lambda^-}\right)^{-1}\left(I-e^{2LP_\lambda^+}\right)^{-1}\Pi_\lambda^{-1}.$$

Remark 8.4. To prove that operator $\mathcal{L}$ generates an analytic semigroup, it is necessary that

$$\frac{4}{7}(\pi-\varepsilon_0)>\frac{\pi}{2},$$

which is true since $\varepsilon_0<\dfrac{\pi}{8}$.

Proof. The invertibility of D_λ relies on the invertibility of $f_\lambda(-Q)$. We will now show that $f_\lambda(-Q)$ is invertible with bounded inverse by using Proposition 5.6.

We recall the following notations for $\lambda\in S_{2(\pi-\varepsilon)/3}$ and $z\in\pi^2+S_{(\pi-\varepsilon)/3}$

$$z_-=z+\frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-}\quad\text{and}\quad z_+=z+\frac{1-\sigma_H^+(1+f_H^+)}{d_H^+};$$

$$\lambda_-=-\frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-}-\frac{\lambda}{d_H^-}\quad\text{and}\quad\lambda_+=-\frac{1-\sigma_H^+(1+f_H^+)}{d_H^+}-\frac{\lambda}{d_H^+};$$

thus $z_\pm\in S_{(\pi-\varepsilon)/3}$ and

$$z-\lambda_-=z_-+\frac{\lambda}{d_H^-}\quad\text{and}\quad z-\lambda_+=z_++\frac{\lambda}{d_H^+}.$$

We consider two cases:

1. When λ verifies: $-\frac{\pi}{3}+\frac{\varepsilon}{3}\leqslant \arg(\lambda)<\frac{2\pi}{3}-\frac{2\varepsilon}{3}$.
   From Proposition 13 in [11], we have

$$-\frac{\pi}{3}+\frac{\varepsilon}{3}\leqslant \arg\left(z_\pm+\frac{\lambda}{d_H^\pm}\right)<\frac{2\pi}{3}-\frac{2\varepsilon}{3},$$

   and

$$-\frac{\pi}{6}+\frac{\varepsilon}{6}\leqslant \arg\left(\sqrt{z-\lambda_\pm}\right)<\frac{\pi}{3}-\frac{\varepsilon}{3},$$

   hence

$$-\frac{\pi}{2}+\frac{\varepsilon}{2}<\arg\left(\frac{\sqrt{z-\lambda_+}}{\sqrt{z-\lambda_-}}\right)=\arg\left(\sqrt{z-\lambda_+}\right)-\arg\left(\sqrt{z-\lambda_-}\right)<\frac{\pi}{2}-\frac{\varepsilon}{2}.$$

   Since

$$\left|\arg\left(\frac{\left(1-e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1+e^{-2L\sqrt{z-\lambda_+}}\right)}{\left(1+e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1-e^{-2L\sqrt{z-\lambda_+}}\right)}\right)\right| \leqslant \left|\arg\left(\frac{\left(1-e^{-2\ell\sqrt{z-\lambda_-}}\right)}{\left(1+e^{-2\ell\sqrt{z-\lambda_-}}\right)}\right)\right| + \left|\arg\left(\frac{\left(1-e^{-2L\sqrt{z-\lambda_+}}\right)}{\left(1+e^{-2L\sqrt{z-\lambda_+}}\right)}\right)\right|,$$

   from Lemma 8.1 statement 2., we have

$$\left|\arg\left(\frac{\left(1-e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1+e^{-2L\sqrt{z-\lambda_+}}\right)}{\left(1+e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1-e^{-2L\sqrt{z-\lambda_+}}\right)}\right)\right|<\frac{2}{3}(\pi-\varepsilon).$$

   For all $z\in\pi^2+S_{(\pi-\varepsilon)/3}$, we note

$$\tilde{f}_\lambda(z)=\frac{\beta_S}{\beta_I}\frac{\sqrt{z-\lambda_+}}{\sqrt{z-\lambda_-}}\frac{\left(1-e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1+e^{-2L\sqrt{z-\lambda_+}}\right)}{\left(1+e^{-2\ell\sqrt{z-\lambda_-}}\right)\left(1-e^{-2L\sqrt{z-\lambda_+}}\right)}=f_\lambda(z)-1.$$

   We deduce, from the previous inequalities, that

$$\left|\arg\left(\tilde{f}_\lambda(z)\right)\right|<\frac{7}{6}(\pi-\varepsilon),$$

   and according to Proposition 5.2, we have

$$|f_\lambda(z)|\geqslant\left(1+\left|\tilde{f}_\lambda(z)\right|\right)\left|\cos\left(\frac{\arg\left(\tilde{f}_\lambda(z)\right)}{2}\right)\right|\geqslant\left|\cos\left(\frac{\arg\left(\tilde{f}_\lambda(z)\right)}{2}\right)\right|.$$

   Now, assume that

$$\varepsilon=\frac{\pi}{7}+\frac{6}{7}\varepsilon_0,$$

   where

$$0<\varepsilon_0<\frac{\pi}{8}.$$

   It follows that

$$|f_\lambda(z)|\geqslant\left|\cos\left(\frac{\arg\left(\tilde{f}_\lambda(z)\right)}{2}\right)\right|>\cos\left(\frac{\pi}{2}-\frac{\varepsilon_0}{2}\right)=\sin\left(\frac{\varepsilon_0}{2}\right)>0.$$

   Therefore $1/f_\lambda$ is bounded and belongs to $H^\infty\left(\pi^2+S_{(\pi-\varepsilon_0)/3}\right)$.
   Finally, using again Proposition 5.6, $f_\lambda(-Q)=\Pi_\lambda$ is boundedly invertible and

$$[f_\lambda(-Q)]^{-1}=\frac{1}{f_\lambda}(-Q).$$

2. When λ verifies: $-\dfrac{2\pi}{3}+\dfrac{2\varepsilon}{3}<\arg(\lambda)\leqslant\dfrac{\pi}{3}-\dfrac{\varepsilon}{3}$.

   By using an analogous method, we obtain the invertibility of Π_λ.

To conclude, there exists a constant $C>0$ independent of λ such that

$$\left\|D_\lambda^{-1}\right\|_{\mathcal{L}(X)}\leqslant C. \tag{29}$$

□

We are now in position to solve system (26). Thanks to Proposition 8.3, D_λ is invertible for $\lambda\in S_{4(\pi-\varepsilon_0)/7}$ and the expressions of constants δ_I and γ_S are given by

$$\left\{\begin{array}{rcl}\delta_I & = & \beta_S\, D_\lambda^{-1}\, P_\lambda^+(P_\lambda^-)^{-1}\left(I+e^{2LP_\lambda^+}\right)(R_S-R_I)\\ & & +D_\lambda^{-1}\left(I-e^{2LP_\lambda^+}\right)\left(\beta_I\, R_I+\beta_S\, P_\lambda^+(P_\lambda^-)^{-1}R_S\right)\\ \\ \gamma_S & = & -\beta_I\, D_\lambda^{-1}\left(I+e^{2\ell P_\lambda^-}\right)(R_S-R_I)\\ & & +D_\lambda^{-1}\left(I-e^{2\ell P_\lambda^-}\right)\left(\beta_I\, R_I+\beta_S\, P_\lambda^+(P_\lambda^-)^{-1}R_S\right).\end{array}\right.$$

We now explicit the solution (h_I,h_S) by using the fact that operators D_λ^{-1}, $P_\lambda^\pm$ and $(P_\lambda^\pm)^{-1}$ commute among themselves.

For $x\in(-\ell,0)$, we have

$$\begin{array}{rcl}h_I(x) & = & 2\,\beta_S\, D_\lambda^{-1}\, P_\lambda^+(P_\lambda^-)^{-1}e^{-xP_\lambda^-}\left(I-e^{2(x+\ell)P_\lambda^-}\right)R_S\\ & & -\beta_S\, D_\lambda^{-1}\, P_\lambda^+(P_\lambda^-)^{-1}e^{-xP_\lambda^-}\left(I-e^{2(x+\ell)P_\lambda^-}\right)\left(I+e^{2LP_\lambda^+}\right)R_I\\ & & +\beta_I\, D_\lambda^{-1}e^{-xP_\lambda^-}\left(I-e^{2(x+\ell)P_\lambda^-}\right)\left(I-e^{2LP_\lambda^+}\right)R_I\\ & & -e^{(x+\ell)P_\lambda^-}w_I(g_I)(-\ell)\\ & & +w_I(g_I)(x),\end{array}$$

where

$$w_I(g_I)(x)=\frac{1}{2}\int_{-\ell}^{x}e^{(x-t)P_\lambda^-}(P_\lambda^-)^{-1}g_I(t)\;dt+\frac{1}{2}\int_{x}^{0}e^{(t-x)P_\lambda^-}(P_\lambda^-)^{-1}g_I(t)\;dt;$$

and for $x\in(0,L)$, we have

$$\begin{array}{rcl}h_S(x) & = & 2\,\beta_I\, D_\lambda^{-1}e^{xP_\lambda^+}\left(I-e^{2(L-x)P_\lambda^+}\right)R_I\\ & & -\beta_I\, D_\lambda^{-1}e^{xP_\lambda^+}\left(I-e^{2(L-x)P_\lambda^+}\right)\left(I+e^{2\ell P_\lambda^-}\right)R_S\\ & & +\beta_S\, D_\lambda^{-1}\, P_\lambda^+(P_\lambda^-)^{-1}e^{xP_\lambda^+}\left(I-e^{2(L-x)P_\lambda^+}\right)\left(I-e^{2\ell P_\lambda^-}\right)R_S\\ & & -e^{(L-x)P_\lambda^+}w_S(g_S)(L)\\ & & +w_S(g_S)(x),\end{array}$$

where

$$w_S(g_S)(x)=\frac{1}{2}\int_{0}^{x}e^{(x-t)P_\lambda^+}(P_\lambda^+)^{-1}g_S(t)\;dt+\frac{1}{2}\int_{x}^{L}e^{(t-x)P_\lambda^+}(P_\lambda^+)^{-1}g_S(t)\;dt,$$

$$R_I=w_I(g_I)(0)-e^{\ell P_\lambda^-}w_I(g_I)(-\ell)=\frac{1}{2}\int_{-\ell}^{0}e^{-tP_\lambda^-}\left(I-e^{2(t+\ell)P_\lambda^-}\right)(P_\lambda^-)^{-1}g_I(t)\;dt,$$

and

$$R_S=w_S(g_S)(0)-e^{LP_\lambda^+}w_S(g_S)(L)=\frac{1}{2}\int_{0}^{L}e^{tP_\lambda^+}\left(I-e^{2(L-t)P_\lambda^+}\right)(P_\lambda^+)^{-1}g_S(t)\;dt.$$

Remark 8.5. Note that in the representation formula of the solution h_I, the first three terms express, in particular, the effect of the transmission conditions between the two habitats; the fourth term expresses the effect of a boundary condition in $-\ell$ and the fifth term, obviously, expresses the effect of the direct and retrograde evolution inside the domain Ω_I.

The same comment is valid for the representation formula of h_S.

8.3 Maximal regularity of the solution

From Proposition 6.1, for all $\eta > 0$, operators $-Q$ and $-Q^{\pm}$ are sectorial of angle η. On the other hand, by Proposition 3.1, p. 191 in [23], we have $-Q \in \mathrm{BIP}\,(X, \eta)$. Moreover, due to (8) and Theorem 2.3, p. 69 in [1], we deduce that $-Q^{\pm} \in \mathrm{BIP}\,(X, \eta)$. Thus, thanks to Theorem 2.4, p. 408 in [29], we obtain that

$$-Q^{\pm} + \frac{\lambda}{d_H^{\pm}} \in \mathrm{BIP}(X, \theta),$$

where

$$\theta = \max\left(\eta, |\arg(\lambda)|\right) = |\arg(\lambda)| < \frac{4(\pi - \varepsilon_0)}{7}.$$

Finally, due to Proposition 3.2.1, e), p. 71 in [18], we deduce that

$$-P_\lambda^{\pm} \in \mathrm{BIP}(X, 2(\pi - \varepsilon_0)/7).$$

Recall that for all fixed $c > 0$, we have

$$\forall\, \psi \in X, \quad e^{cP_\lambda^{\pm}} \psi \in D((P_\lambda^{\pm})^{\infty}) = \bigcap_{k \geqslant 0} D((P_\lambda^{\pm})^{k}) = D(Q^{\infty}). \tag{30}$$

For the maximal regularity of h_I, we have to show that

$$h_I \in W^{2,p}(-\ell, 0; X) \cap L^p\left(-\ell, 0; D\left((P_\lambda^-)^2\right)\right).$$

It suffices to show, for instance, that

$$h_I \in L^p\left(-\ell, 0; D\left((P_\lambda^-)^2\right)\right).$$

We recall that, for almost every $x \in (-\ell, 0)$, we have

$$\begin{array}{rcl} h_I(x) & = & 2\,\beta_S\, D_\lambda^{-1}\, P_\lambda^{+} (P_\lambda^-)^{-1} e^{-xP_\lambda^-} \left(I - e^{2(x+\ell)P_\lambda^-}\right) R_S \\ & & -\beta_S\, D_\lambda^{-1}\, P_\lambda^{+} (P_\lambda^-)^{-1} e^{-xP_\lambda^-} \left(I - e^{2(x+\ell)P_\lambda^-}\right)\left(I + e^{2LP_\lambda^+}\right) R_I \\ & & +\beta_I\, D_\lambda^{-1} e^{-xP_\lambda^-} \left(I - e^{2(x+\ell)P_\lambda^-}\right)\left(I - e^{2LP_\lambda^+}\right) R_I \\ & & -e^{(x+\ell)P_\lambda^-} w_I(g_I)(-\ell) \\ & & +w_I(g_I)(x). \end{array}$$

The last term is directly treated using Theorem 5.11, thus

$$w_I(g_I) \in L^p\left(-\ell, 0; D\left((P_\lambda^-)^2\right)\right).$$

Moreover, we have

$$\begin{array}{rcl} e^{(x+\ell)P_\lambda^-} w_I(g_I)(-\ell) & = & \dfrac{1}{2} e^{(x+\ell)P_\lambda^-} \displaystyle\int_{-\ell}^{0} e^{(s+\ell)P_\lambda^-} (P_\lambda^-)^{-1} g(s)\; ds \\ & = & \dfrac{1}{2} (P_\lambda^-)^{-1} \displaystyle\int_{-\ell}^{x} e^{(x-s)P_\lambda^-} e^{2(s+\ell)P_\lambda^-} g(s)\; ds \\ & & +\dfrac{1}{2} (P_\lambda^-)^{-1} e^{2(x+\ell)P_\lambda^-} \displaystyle\int_{x}^{0} e^{(s-x)P_\lambda^-} g(s)\; ds. \end{array}$$

Since

$$s \longmapsto e^{2(s+\ell)P_\lambda^-} g(s) \in L^p(-\ell, 0; X),$$

using again Theorem 5.11, we obtain that

$$x \longmapsto e^{(x+\ell)P_\lambda^-} w_I(g_I)(-\ell) \in L^p\left(-\ell, 0; D\left((P_\lambda^-)^2\right)\right).$$

Let us, for instance, analyze the following first term, since all the other terms can be treated similarly:

$$\begin{aligned}
&2\,\beta_S\, D_\lambda^{-1}\, P_\lambda^+ (P_\lambda^-)^{-1} e^{-xP_\lambda^-}\left(I - e^{2(x+\ell)P_\lambda^-}\right) R_S \\
&= 2\,\beta_S\, D_\lambda^{-1}\, P_\lambda^+ (P_\lambda^-)^{-1} e^{-xP_\lambda^-}\left(I - e^{2(x+\ell)P_\lambda^-}\right) w_S(g_S)(0) \\
&\quad -2\,\beta_S\, D_\lambda^{-1}\, P_\lambda^+ (P_\lambda^-)^{-1} e^{-xP_\lambda^-}\left(I - e^{2(x+\ell)P_\lambda^-}\right) e^{LP_\lambda^+} w_S(g_S)(L).
\end{aligned}$$

From (30), we just have to study

$$\begin{aligned}
&\left(P_\lambda^-\right)^2 D_\lambda^{-1}\, P_\lambda^+ (P_\lambda^-)^{-1} e^{-xP_\lambda^-}\left(I - e^{2(x+\ell)P_\lambda^-}\right) w_S(g_S)(0) \\
&= \frac{1}{2}\, D_\lambda^{-1}\left(I - e^{2(x+\ell)P_\lambda^-}\right) P_\lambda^- e^{-xP_\lambda^-} \int_0^L e^{tP_\lambda^+} g_S(t)\, dt.
\end{aligned}$$

According to Lemma 18, p. 19 in [11], we have

$$\left\{\begin{array}{ll}
\displaystyle\int_0^L e^{tP_\lambda^+} g_S(t)\, dt & \in \left(D(P_\lambda^+), X\right)_{\frac{1}{p},p} \\
\displaystyle\int_0^L e^{(L-t)P_\lambda^+} g_S(t)\, dt & \in \left(D(P_\lambda^+), X\right)_{\frac{1}{p},p} \\
\displaystyle\int_{-\ell}^0 e^{-tP_\lambda^-} g_I(t)\, dt & \in \left(D(P_\lambda^-), X\right)_{\frac{1}{p},p} \\
\displaystyle\int_{-\ell}^0 e^{(t+\ell)P_\lambda^-} g_I(t)\, dt & \in \left(D(P_\lambda^-), X\right)_{\frac{1}{p},p}.
\end{array}\right. \tag{31}$$

Since $D(P_\lambda^+) = D(P_\lambda^-) = D(\sqrt{-Q})$, it follows that

$$\left(D(P_\lambda^+), X\right)_{\frac{1}{p},p} = \left(D(P_\lambda^-), X\right)_{\frac{1}{p},p} = \left(D\left(\sqrt{-Q}\right), X\right)_{\frac{1}{p},p} = (D(Q), X)_{\frac{1}{2}+\frac{1}{2p},p}.$$

Thus, due to (15), for $n = 1$, we deduce that

$$x \longmapsto P_\lambda^- e^{-xP_\lambda^-} \int_0^L e^{tP_\lambda^+} g_S(t)\, dt \in L^p(-\ell, 0; X);$$

hence

$$h_I \in L^p\left(-\ell, 0; D\left((P_\lambda^-)^2\right)\right).$$

In the same way, we obtain that

$$h_S \in W^{2,p}(0, L; X) \cap L^p\left(0, L; D\left((P_\lambda^+)^2\right)\right).$$

9 Estimation of the norm of the resolvent operator

Lemma 9.1. Let T be a linear operator such that $-T \in \mathrm{BIP}(X, \theta_T)$, with $\theta_T \in [0, \pi/2[$ and $0 \in \rho(T)$. Let $g \in L^p(a, b; X)$ with $1 < p < +\infty$ and $a < b$. For all $\theta \in]0, \pi - \theta_T[$, $\mu \in \overline{S_\theta} \subset \rho(-T)$ and $x \in [a, b]$, we set

$$I_{\mu,g}(x) = \int_a^x e^{-(x-s)\sqrt{-T+\mu I}} g(s)\, ds \quad \text{and} \quad J_{\mu,g}(x) = \int_x^b e^{-(s-x)\sqrt{-T+\mu I}} g(s)\, ds.$$

Then, we have

$$\|I_{\mu,f}\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{1+|\mu|}}\, \|f\|_{L^p(a,b;X)} \quad \text{and} \quad \|J_{\mu,f}\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{1+|\mu|}}\, \|f\|_{L^p(a,b;X)},$$

where $C > 0$ is independent of g and μ.

This result is a consequence of Lemma 4.6 in [15] and Lemma 4.11 in [22].

Lemma 9.2. Let $g \in L^p(a,b;X)$, $1 < p < +\infty$ and T be a linear operator such that $-T \in \mathrm{BIP}(X,\theta_T)$, where $\theta_T \in [0,\pi/2[$ and $0 \in \rho(T)$. Let $\theta \in]0, \pi-\theta_T[$ fixed. Then, there exists $C>0$, such that, for all $\eta, \mu \in \overline{S_\theta} \subset \rho(-T)$, we have

1. $$\left\| e^{-(.-a)\sqrt{-T+\eta I}} \int_a^b e^{-(s-a)\sqrt{-T+\mu I}} g(s)\, ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|g\|_{L^p(a,b;X)},$$
2. $$\left\| e^{-(.-a)\sqrt{-T+\eta I}} \int_a^b e^{-(b-s)\sqrt{-T+\mu I}} g(s)\, ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|g\|_{L^p(a,b;X)},$$
3. $$\left\| e^{-(b-.)\sqrt{-T+\eta I}} \int_a^b e^{-(b-s)\sqrt{-T+\mu I}} g(s)\, ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|g\|_{L^p(a,b;X)},$$
4. $$\left\| e^{-(b-.)\sqrt{-T+\eta I}} \int_a^b e^{-(s-a)\sqrt{-T+\mu I}} g(s)\, ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|g\|_{L^p(a,b;X)}.$$

This Lemma is proved in [22], Lemma 4.12, p. 33.

Remark 9.3. In our case, $T = Q^+$ or $T = Q^-$; likewise η and μ will be replaced by λ/d_H^- and λ/d_H^+.

According to (17), there exists $C>0$, such that for all $\lambda \in S_{12(\pi-\varepsilon_0)/21}$

$$\left\|(P_\lambda^-)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{\sqrt{1+|\lambda|}} \quad \text{and} \quad \left\|(P_\lambda^+)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{\sqrt{1+|\lambda|}}. \tag{32}$$

Moreover, for $\alpha \in \mathbb{R}$ and $t_0 > 0$ fixed, due to [14], Lemma 2.6, (b), p. 104, there exist $K, C, \omega > 0$, such that

$$\left\|(-P_\lambda^\pm)^\alpha\, e^{t_0 P_\lambda^\pm}\right\|_{\mathcal{L}(X)} \leqslant K e^{-t_0\omega\sqrt{|\lambda|/d_H^\pm}} \leqslant K e^{-C\sqrt{|\lambda|}}. \tag{33}$$

Lemma 9.4. There exists $C>0$ such that for all $\lambda \in S_{4(\pi-\varepsilon_0)/7}$

$$\|P_\lambda^-(P_\lambda^+)^{-1}\|_{\mathcal{L}(X)} \leqslant C \quad \text{and} \quad \|P_\lambda^+(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)} \leqslant C.$$

Proof. We have

$$\begin{aligned} P_\lambda^-(P_\lambda^+)^{-1} &= (P_\lambda^-)^2(P_\lambda^-)^{-1}(P_\lambda^+)^{-1} = \left(-Q^- + \frac{\lambda}{d_H^-} I\right)(P_\lambda^-)^{-1}(P_\lambda^+)^{-1} \\ &= -Q(P_\lambda^-)^{-1}(P_\lambda^+)^{-1} + \left(\frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} + \frac{\lambda}{d_H^-}\right)(P_\lambda^-)^{-1}(P_\lambda^+)^{-1} \\ &= \sqrt{-Q}(P_\lambda^-)^{-1}\sqrt{-Q}(P_\lambda^+)^{-1} + \left(\frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-} + \frac{\lambda}{d_H^-}\right)(P_\lambda^-)^{-1}(P_\lambda^+)^{-1}, \end{aligned}$$

hence

$$\begin{aligned} \|P_\lambda^-(P_\lambda^+)^{-1}\|_{\mathcal{L}(X)} &\leqslant \|\sqrt{-Q}(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)}\|\sqrt{-Q}(P_\lambda^+)^{-1}\|_{\mathcal{L}(X)} \\ &\quad + \left(\frac{|1-\sigma_H^-(1+\nu f_H^-)|}{d_H^-} + \frac{|\lambda|}{d_H^-}\right)\|(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)}\|(P_\lambda^+)^{-1}\|_{\mathcal{L}(X)}. \end{aligned}$$

From (32), there exists $C>0$, independent of λ, such that

$$\frac{|1-\sigma_H^-(1+\nu f_H^-)|}{d_H^-}\|(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)}\|(P_\lambda^+)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C}{1+|\lambda|} \leqslant C,$$

and

$$\frac{|\lambda|}{d_H^-}\,\|(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)}\|(P_\lambda^+)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C\,|\lambda|}{1+|\lambda|} \leqslant C.$$

We have

$$\sqrt{-Q}(P_\lambda^-)^{-1} = -(-Q)^{1/2}\left(-Q + \frac{1-\sigma_H^-(1+\nu f_H^-)}{d_H^-}I + \frac{\lambda}{d_H^-}I\right)^{-1/2},$$

and

$$\sqrt{-Q}(P_\lambda^+)^{-1} = -(-Q)^{1/2}\left(-Q + \frac{1-\sigma_H^+(1+f_H^+)}{d_H^+}I + \frac{\lambda}{d_H^+}I\right)^{-1/2},$$

then using [14], Lemma 2.6, (a), p. 104, there exists $C>0$ such that

$$\|\sqrt{-Q}(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)} \leqslant C \quad \text{and} \quad \|\sqrt{-Q}(P_\lambda^+)^{-1}\|_{\mathcal{L}(X)} \leqslant C.$$

In the same way, we obtain that

$$\|P_\lambda^+(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)} \leqslant C.$$

□

We recall that the solution h_I is given by:

$$\begin{aligned} h_I(x) \;=\; & 2\,\beta_S\,D_\lambda^{-1}\,P_\lambda^+(P_\lambda^-)^{-1}e^{-xP_\lambda^-}\left(I-e^{2(x+\ell)P_\lambda^-}\right)R_S \\ & -\beta_S\,D_\lambda^{-1}\,P_\lambda^+(P_\lambda^-)^{-1}e^{-xP_\lambda^-}\left(I-e^{2(x+\ell)P_\lambda^-}\right)\left(I+e^{2LP_\lambda^+}\right)R_I \\ & +\beta_I\,D_\lambda^{-1}e^{-xP_\lambda^-}\left(I-e^{2(x+\ell)P_\lambda^-}\right)\left(I-e^{2LP_\lambda^+}\right)R_I \\ & -e^{(x+\ell)P_\lambda^-}w_I(g_I)(-\ell) \\ & +w_I(g_I)(x), \end{aligned}$$

where

$$w_I(g_I)(x) = \frac{1}{2}(P_\lambda^-)^{-1}\int_{-\ell}^{x} e^{(x-t)P_\lambda^-}g_I(t)\;dt + \frac{1}{2}(P_\lambda^-)^{-1}\int_{x}^{0} e^{(t-x)P_\lambda^-}g_I(t)\;dt,$$

and

$$\left\{\begin{aligned} R_I &= \frac{1}{2}(P_\lambda^-)^{-1}\int_{-\ell}^{0} e^{-tP_\lambda^-}\left(I-e^{2(t+\ell)P_\lambda^-}\right)g_I(t)\;dt \\ R_S &= \frac{1}{2}(P_\lambda^+)^{-1}\int_{0}^{L} e^{tP_\lambda^+}\left(I-e^{2(L-t)P_\lambda^+}\right)g_S(t)\;dt. \end{aligned}\right.$$

Since P_λ^+ and P_λ^- generate bounded analytic semigroups, there exists $C>0$, independent of λ, such that

$$\left\|I-e^{2(x+\ell)P_\lambda^-}\right\|_{\mathcal{L}(X)} \leqslant C, \quad \left\|I-e^{2LP_\lambda^+}\right\|_{\mathcal{L}(X)} \leqslant C \quad \text{and} \quad \left\|I+e^{2LP_\lambda^+}\right\|_{\mathcal{L}(X)} \leqslant C.$$

Then, for almost every $x\in(-\ell,0)$, we have

$$\begin{aligned} \|h_I(x)\|_X \;\leqslant\; & 2\,\beta_S\|D_\lambda^{-1}\|_{\mathcal{L}(X)}\|P_\lambda^+(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)}\left\|I-e^{2(x+\ell)P_\lambda^-}\right\|_{\mathcal{L}(X)}\|e^{-xP_\lambda^-}R_S\|_X \\ & +\beta_S\,\|D_\lambda^{-1}\|_{\mathcal{L}(X)}\|P_\lambda^+(P_\lambda^-)^{-1}\|_{\mathcal{L}(X)}\left\|I-e^{2(x+\ell)P_\lambda^-}\right\|_{\mathcal{L}(X)}\left\|I+e^{2LP_\lambda^+}\right\|_{\mathcal{L}(X)}\|e^{-xP_\lambda^-}R_I\|_X \\ & +\beta_I\|D_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|I-e^{2(x+\ell)P_\lambda^-}\right\|_{\mathcal{L}(X)}\left\|I-e^{2LP_\lambda^+}\right\|_{\mathcal{L}(X)}\|e^{-xP_\lambda^-}R_I\|_X \\ & +\|e^{(x+\ell)P_\lambda^-}w_I(g_I)(-\ell)\|_X + \|w_I(g_I)(x)\|_X \\ \leqslant\; & C\left(\|e^{-xP_\lambda^-}R_S\|_X + \|e^{-xP_\lambda^-}R_I\|_X\right) + \|e^{(x+\ell)P_\lambda^-}w_I(g_I)(-\ell)\|_X + \|w_I(g_I)(x)\|_X; \end{aligned}$$

hence

$$\begin{array}{rcl}\|h_I\|_{L^p(-\ell,0;X)} & \leqslant & C\left(\|e^{-\cdot P_\lambda^-}R_I\|_{L^p(-\ell,0;X)} + \|e^{-\cdot P_\lambda^-}R_S\|_{L^p(-\ell,0;X)}\right) \\ & & +\|e^{(\cdot+\ell)P_\lambda^-}w_I(g_I)(-\ell)\|_{L^p(-\ell,0;X)} + \|w_I(g_I)(\cdot)\|_{L^p(-\ell,0;X)}.\end{array}$$

From (32) and Lemma 9.2, there exists $C>0$, independent of λ, such that

$$\begin{array}{rcl}\|e^{-\cdot P_\lambda^-}R_I\|_{L^p(-\ell,0;X)} & \leqslant & \dfrac{C}{\sqrt{1+|\lambda|}}\left\|e^{-\cdot P_\lambda^-}\displaystyle\int_{-\ell}^0 e^{-tP_\lambda^-}\left(I-e^{2(t+\ell)P_\lambda^-}\right)g_I(t)\;dt\right\|_{L^p(-\ell,0;X)} \\ & \leqslant & \dfrac{C}{1+|\lambda|}\left\|\left(I-e^{2(\cdot+\ell)P_\lambda^-}\right)g_I\right\|_{L^p(-\ell,0;X)} \\ & \leqslant & \dfrac{C}{1+|\lambda|}\|g_I\|_{L^p(-\ell,0;X)}.\end{array}$$

Similarly, we have

$$\begin{array}{rcl}\|e^{-\cdot P_\lambda^-}R_S\|_{L^p(-\ell,0;X)} & \leqslant & \dfrac{C}{\sqrt{1+|\lambda|}}\left\|e^{-\cdot P_\lambda^-}\displaystyle\int_0^L e^{tP_\lambda^+}\left(I-e^{2(L-t)P_\lambda^+}\right)g_S(t)\;dt\right\|_{L^p(0,L;X)} \\ & \leqslant & \dfrac{C}{1+|\lambda|}\left\|\left(I-e^{2(L-\cdot)P_\lambda^+}\right)g_S\right\|_{L^p(0,L;X)} \\ & \leqslant & \dfrac{C}{1+|\lambda|}\|g_S\|_{L^p(0,L;X)},\end{array}$$

and

$$\|e^{(\cdot+\ell)P_\lambda^-}w_I(g_I)(-\ell)\|_{L^p(-\ell,0;X)} \leqslant \frac{C}{1+|\lambda|}\|g_I\|_{L^p(-\ell,0;X)}.$$

Then, due to (32) and Lemma 9.1, there exists $C>0$, independent of λ, such that

$$\|w_I(g_I)(\cdot)\|_{L^p(-\ell,0;X)} \leqslant \frac{C}{1+|\lambda|/d_H^-}\|g_I\|_{L^p(-\ell,0;X)} \leqslant \frac{C}{1+|\lambda|}\|g_I\|_{L^p(-\ell,0;X)}.$$

Finally, there exists $C>0$ such that, for all $\lambda\in S_{4(\pi-\varepsilon_0)/7}$, we obtain

$$\|h_I\|_{L^p(-\ell,0;X)} \leqslant \frac{C}{1+|\lambda|}\left(\|g_I\|_{L^p(-\ell,0;X)} + \|g_S\|_{L^p(0,L;X)}\right),$$

and

$$\|h_S\|_{L^p(0,L;X)} \leqslant \frac{C}{1+|\lambda|}\left(\|g_I\|_{L^p(-\ell,0;X)} + \|g_S\|_{L^p(0,L;X)}\right);$$

hence

$$\begin{array}{rcl}\|h\|_{L^p(\Omega)} & = & \|h_I\|_{L^p(\Omega_I)} + \|h_S\|_{L^p(\Omega_S)} \\ & = & \|h_I\|_{L^p(-\ell,0;X)} + \|h_S\|_{L^p(0,L;X)} \\ & \leqslant & \dfrac{C}{1+|\lambda|}\left(\|g_I\|_{L^p(-\ell,0;X)} + \|g_S\|_{L^p(0,L;X)}\right) \\ & \leqslant & \dfrac{C}{1+|\lambda|}\left(\|n_I\|_{L^p(-\ell,0;X)} + \|n_S\|_{L^p(0,L;X)}\right) \\ & \leqslant & \dfrac{C}{1+|\lambda|}\left(\|n_I\|_{L^p(\Omega_I)} + \|n_S\|_{L^p(\Omega_S)}\right) \\ & \leqslant & \dfrac{C}{1+|\lambda|}\|n\|_{L^p(\Omega)}.\end{array}$$

The same techniques are used, without assumption (8), in order to obtain the estimates concerning the juveniles and the adults, that is

$$\|j\|_{L^p(\Omega)} = \|j_I\|_{L^p(\Omega_I)} + \|j_S\|_{L^p(\Omega_S)} \leqslant \frac{C}{1+|\lambda|}\|k\|_{L^p(\Omega)},$$

and

$$\|a\|_{L^p(\Omega)} = \|a_I\|_{L^p(\Omega_I)} + \|a_S\|_{L^p(\Omega_S)} \leqslant \frac{C}{1+|\lambda|}\|m\|_{L^p(\Omega)}.$$

Therefore, in $\mathcal{E} = (L^p(\Omega))^3$, we conclude that

$$\left\| \begin{pmatrix} j \\ a \\ h \end{pmatrix} \right\|_{\mathcal{E}} = \max\left(\|j\|_{L^p(\Omega)}, \|a\|_{L^p(\Omega)}, \|h\|_{L^p(\Omega)}\right) \leqslant \frac{C}{1+|\lambda|} \left\| \begin{pmatrix} k \\ m \\ n \end{pmatrix} \right\|_{\mathcal{E}},$$

which complete the proof of Theorem 3.5.

Furthermore, it would be interesting to study a system equivalent to (3) with density dependance using , for example, Ricker recruitment function for hosts.
Regarding the vectors, no condition is required on them. And it would be interesting to add density dependence.

10 Conclusion

In this work, we have modeled the sylvatic transmission of Chagas disease by incorporating vertical transmission within host populations. The primary originality of this research lies in considering, for the first time, that a healthy individual (vector or host) can move from an infected area toward a safe one via a Brownian motion, which better reflects field observations. The constructed system of reaction-diffusion equations was formulated as the following evolution equation:

$$\begin{cases} V'(t) &= (\mathcal{L}+\mathcal{B})V(t) \\ V(0) &= V_0, \end{cases} \tag{34}$$

for a suitable initial condition V_0.

From a mathematical standpoint, the originality of this work is the generation of an analytic semigroup by $\mathcal{L}$ (see Theorem 3.5 and the link with sectoriality in Definition 5.4), specifically proving that $-\mathcal{L}$ is sectorial with an angle strictly less than $\pi/2$. This result, along with the satisfaction of condition (8), depends solely on demographic parameters, spatial dispersion, and the vertical transmission rate of the hosts. Notably, if the host population is density-dependent, condition (8) is naturally satisfied.

Establishing these properties, particularly the invertibility of the determinant D_λ defined in section 8.1, required several advanced mathematical tools:

- the H^∞-calculus,
- specific analytic semigroup properties,
- the notion of BIP operators,
- real interpolation space theory.

Using the Kato's perturbation theory, see [19], Chapter 4, p. 189 and Theorem 2.4, p. 499, we can prove that $\mathcal{L}+\mathcal{B}$ generates an analytic semigroup.
Note that $\mathcal{E} = (L^p(\Omega))^3$ is a UMD space, see [4] and [5], which implies that $L^p(0,T;\mathcal{E})$, $p \in (1,+\infty)$ is also a UMD space. On the other hand, we can also show the BIP character of $-(\mathcal{L}+\mathcal{B})$ in $\mathcal{E}$.
Finally, these results allow the application of the Dore-Venni sum theory [13] to achieve the complete resolution of the evolution problem (34). It should be emphasized, however, that the numerical exploitation of model (3) remains contingent upon the subsequent proof of uniqueness and regularity of the solution.

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