Generalized linear models for population dynamics in two juxtaposed habitats

R. Labbas, K. Lemrabet, S. Maingot & A. Thorel*

R. L., S. M. & A. T.: Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, 76600 Le Havre, France.

K. L.: USTHB, LAMNEDP, Faculté de Mathématiques, BP.32, El Alia, Bab Ezzouar, 16111 Alger, Algérie.

Abstract

In this work we introduce a generalized linear model regulating the spread of population displayed in a d-dimensional spatial region Ω of $\mathbb{R}^d$ constituted by two juxtaposed habitats having a common interface Γ. This model is described by an operator $\mathcal{L}$ of fourth order combining the Laplace and Biharmonic operators under some natural boundary and transmission conditions. We then invert explicitly this operator in L^p-spaces using the H^∞-calculus and the Dore-Venni sums theory. This main result will lead us in a later work to study the nature of the semigroup generated by $\mathcal{L}$ which is important for the study of the complete nonlinear generalized diffusion equation associated to it.

Key Words and Phrases: Population dynamics, Diffusion equation, Semigroups, Landau-Ginzburg free energy functional.

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1 Introduction

The partial differential equations play a natural role in population dynamics, in particular in the reaction-diffusion models which are derived from the well known Fick's law.

An important problem in population ecology is the effect of environmental changes on the growth and diffusion of the species in areas made up of various habitats. In this situation and in order to understand how populations interact between the habitats, it is necessary to have spatially explicit models incorporating individual behaviour at different boundaries and interfaces of the habitats.

If $u(t,.)$ denotes the population density, the classical Fickian equations in each habitat for these models are typically of the form

$$\frac{\partial u}{\partial t} = l\Delta u + F(u),$$

where F is the nonlinear growth interaction and l is the positive coefficient diffusion (which can be variable).

The variety and the complexity of the habitats and the individuals are not well modeled by spatial effects to be simply Fickian diffusion (as, for example, models of cell motion). An approach based on a the Landau-Ginzburg free energy functional and on the variational derivative consider the more generalized following diffusion equation for growth and dispersal in a population

$$\frac{\partial u}{\partial t} = -k\Delta^2 u + l\Delta u + F(u),$$

where k is generally positive and l is a number which can be negative, see [4], p. 238. In this paper, we only consider the case when k and l are positive, but our techniques can be extended to $k, l \in \mathbb{R} \setminus \{0\}$, satisfying some conditions, this will be done in a forthcoming paper.

Now consider the d-area $\Omega = \Omega_- \cup \Omega_+$ constituted by the two juxtaposed habitats

$$\left\{\begin{array}{lcl} \Omega_- & := & (a,\gamma) \times \omega \\ \Omega_+ & := & (\gamma, b) \times \omega, \end{array}\right.$$

with their interface

$$\Gamma = \{\gamma\} \times \omega,$$

where $a, \gamma, b \in \mathbb{R}$ with $a < \gamma < b$ and ω being an open bounded regular set of $\mathbb{R}^{d-1}$. Consider the following linear stationary dispersal equations

$$(EQ_{pde}) \left\{\begin{array}{lll} -k_- \Delta^2 u_- + l_- \Delta u_- = f_- & \text{in} & \Omega_- \\ -k_+ \Delta^2 u_+ + l_+ \Delta u_+ = f_+ & \text{in} & \Omega_+, \end{array}\right.$$

where

$$u = \left\{\begin{array}{lll} u_- & \text{in} & \Omega_- \\ u_+ & \text{in} & \Omega_+ \end{array}\right. \quad \text{and} \quad f = \left\{\begin{array}{lll} f_- & \text{in} & \Omega_- \\ f_+ & \text{in} & \Omega_+, \end{array}\right.$$

with f given in $L^p(a, b; L^p(\omega)) = L^p(\Omega)$, and $k_\pm, l_\pm$ are positive numbers. The spatial variables will be denoted by (x, y), $x \in (a, b)$ and $y \in \omega$. The above equations will be considered under the following boundary and transmission conditions

$$(BC_{pde}) \left\{\begin{array}{l} (1) \left\{\begin{array}{ll} u_-(x,\zeta) = 0, \; x \in (a,\gamma), \; \zeta \in \partial\omega & u_+(x,\zeta) = 0, \; x \in (\gamma, b), \; \zeta \in \partial\omega \\ \Delta u_-(x,\zeta) = 0, \; x \in (a,\gamma), \; \zeta \in \partial\omega & \Delta u_+(x,\zeta) = 0, \; x \in (\gamma, b), \; \zeta \in \partial\omega \end{array}\right. \\ (2) \left\{\begin{array}{lclclcl} u_-(a,y) & = & \varphi_1^-(y), & u_+(b,y) & = & \varphi_1^+(y), & y \in \omega \\ \dfrac{\partial u_-}{\partial x}(a,y) & = & \varphi_2^-(y), & \dfrac{\partial u_+}{\partial x}(b,y) & = & \varphi_2^+(y), & y \in \omega, \end{array}\right. \end{array}\right.$$

($\varphi_1^\pm$ and $\varphi_2^\pm$ will be given in appropriated spaces) and

$$(TC_{pde}) \left\{\begin{array}{rcll} u_- & = & u_+ & \text{on } \Gamma \\ \dfrac{\partial u_-}{\partial x} & = & \dfrac{\partial u_+}{\partial x} & \text{on } \Gamma \\ k_- \Delta u_- & = & k_+ \Delta u_+ & \text{on } \Gamma \\ \dfrac{\partial}{\partial x}\left(k_- \Delta u_- - l_- u_-\right) & = & \dfrac{\partial}{\partial x}\left(k_+ \Delta u_+ - l_+ u_+\right) & \text{on } \Gamma. \end{array}\right.$$

Now, define the following homogeneous dispersal linear operator

$$\left\{\begin{array}{l} D(\mathcal{L}) = \{u \in L^p(\Omega) : \Delta u_\pm, \Delta^2 u_\pm \in L^p(\Omega_\pm) \text{ and } u_\pm \text{ satisfies } (BC_0) \text{ and } (TC_{pde})\} \\ \mathcal{L}u = \left\{\begin{array}{lll} -k_- \Delta^2 u_- + l_- \Delta u_- & \text{in} & \Omega_- \\ -k_+ \Delta^2 u_+ + l_+ \Delta u_+ & \text{in} & \Omega_+, \end{array}\right. \end{array}\right.$$

where (BC_0) corresponds to (BC_{pde}) with $\varphi_1^+ = \varphi_1^- = \varphi_2^+ = \varphi_2^- = 0$.

Therefore, in this work, we will focus ourselves on proving essentially the invertibility of $\mathcal{L}$; this study will be very useful to analyze the following spectral equation

$$\mathcal{L}u - \lambda u = f, \quad \lambda \in \mathbb{C},$$

in order to characterize the nature of the semigroup generated by $\mathcal{L}$. On the other hand the same techniques used here will apply for this analysis. We know the importance of this property in the study of the generalized diffusion complete equation quoted above.

Let us comment on the boundary and transmission conditions.

The first boundary conditions of (1) in (BC_{pde}) simply mean that the individuals die when they reach on the other parts of the boundaries $(a, b) \times \partial\omega$ (which means that we have an inhospitable border); the second of (1) mean that there is no dispersal in the normal direction. We deduce that the dispersal vanishes on $(a,b)\times\partial\omega$, that is $\Delta u_- = 0$ on $(a,\gamma)\times\partial\omega$ and $\Delta u_+ = 0$ on $(\gamma,b)\times\partial\omega$.

In (2) of (BC_{pde}), the population density and the flux are given, for instance on $\{a\}\times\omega$ and on $\{b\}\times\omega$. This signifies that the habitats are not segregated.

In (TC_{pde}), the two first transmission conditions mean the continuity of the density and its flux at the interface, while the two second express the continuity of the dispersal and its flux (in some sense) at Γ.

We can consider more realistic transmission conditions with the noncontinuity of the density and the flux but including the continuity of the generalized dispersal:

$$-k_-\Delta^2 u_- + l_-\Delta u_- = -k_+\Delta^2 u_+ + l_+\Delta u_+ \quad \text{on} \quad \Gamma.$$

This situation requires to work in spaces built on the continuous functions. We will consider this case in a future work. Note that, when we consider different types of habitats, the response of individuals at the interface is important for the overall movement behaviour.

In many works, a generalized diffusion model is considered. Let us quote a number of them.

In [4] and in [18], the authors have presented in one dimensional case a nonlinear model with spatial structure characterized by a fourth order operator in only one habitat. They used essentially a Landau-Ginzburg free energy functional.

We were essentially inspired by these works to deduce a linear d-dimensional model set in two bounded juxtaposed cylindrical habitats which requires necessarily boundary and transmission conditions.

We will then base ourselves on similar techniques to those used in the works of [7] and [8].

The paper is organized as follows. First, in section 2, we present the PDE transmission problem (P_{pde}) and with the help of operator A_0 defined below, we give its operational writing. We will then study problem (P) with a general operator A instead of A_0.

Then, in section 3, we recall what is a BIP operator, we precise our notations about interpolation spaces, we set our hypotheses and their consequences. We explain how to solve our problem (P) by introducing two auxiliary problems (P_-) and (P_+). We then present our main result in Theorem 3.4. As a consequence of this theorem, we obtain the Corollary 3.6 which states existence and uniqueness of the solution of problem $(EQ_{pde}) - (BC_{pde}) - (TC_{pde})$ quoted above.

In section 4, we give technical results which help us to prove our main result. In Proposition 4.2 and in Proposition 4.4 we solve problems (P_-) and (P_+) provided that the data are in some real interpolation spaces. We establish (see Theorem 4.6) a useful technical result which allows us to prove Theorem 3.4. Then, we show some technical lemmas which lead us to apply functional calculus.

Section 5 is devoted to the proof of Theorem 3.4. This section is composed of three parts: in the first part, we use Theorem 4.6 to explicit the determinant of the transmission system. In the second part, we inverse the determinant of the transmission system using functional calculus. Finally, in the last part, we show that the general transmission problem has a unique classical solution by establishing the regularity of this solution.

2 Operational formulation

Consider now the problem

$$
(P_{pde}) \begin{cases}
k_-\Delta^2 u_- - l_-\Delta u_- = f_- \quad \text{in } \Omega_- \\
k_+\Delta^2 u_+ - l_+\Delta u_+ = f_+ \quad \text{in } \Omega_+ \\
u_-(x,\zeta) = 0, \quad x \in (a,\gamma),\ \zeta \in \partial\omega, \quad u_+(x,\zeta) = 0, \quad x \in (\gamma,b),\ \zeta \in \partial\omega \\
\Delta u_-(x,\zeta) = 0, \quad x \in (a,\gamma),\ \zeta \in \partial\omega, \quad \Delta u_+(x,\zeta) = 0, \quad x \in (\gamma,b),\ \zeta \in \partial\omega \\
u_-(a,y) = \varphi_1^-(y), \quad y \in \omega, \quad u_+(b,y) = \varphi_1^+(y), \quad y \in \omega \\
\dfrac{\partial u_-}{\partial x}(a,y) = \varphi_2^-(y), \quad y \in \omega, \quad \dfrac{\partial u_+}{\partial x}(b,y) = \varphi_2^+(y), \quad y \in \omega \\
u_- = u_+ \quad \text{on } \Gamma \\
\dfrac{\partial u_-}{\partial x} = \dfrac{\partial u_+}{\partial x} \quad \text{on } \Gamma \\
k_-\Delta u_- = k_+\Delta u_+ \quad \text{on } \Gamma \\
\dfrac{\partial}{\partial x}\left(k_-\Delta u_- - l_- u_-\right) = \dfrac{\partial}{\partial x}\left(k_+\Delta u_+ - l_+ u_+\right) \quad \text{on } \Gamma.
\end{cases}
$$

Let us define A_0, the Laplace operator in $\mathbb{R}^{d-1}$, $d \in \mathbb{N} \setminus \{0,1\}$, as follows

$$
\begin{cases}
D(A_0) := \{\psi \in W^{2,p}(\omega) : \psi = 0 \ on\ \partial\omega\} \\
\forall \psi \in D(A_0), \quad A_0\psi = \Delta_y \psi.
\end{cases} \tag{1}
$$

Thus, using operator A_0, problem (P_{pde}) becomes

$$
\begin{cases}
u_-^{(4)}(x) + (2A_0 - \dfrac{l_-}{k_-} I)u_-''(x) + (A_0^2 - \dfrac{l_-}{k_-} A_0)u_-(x) = f_-(x), \quad \text{for a.e. } x \in (a,\gamma) \\
u_+^{(4)}(x) + (2A_0 - \dfrac{l_+}{k_+} I)u_+''(x) + (A_0^2 - \dfrac{l_+}{k_+} A_0)u_+(x) = f_+(x), \quad \text{for a.e. } x \in (\gamma,b) \\
u_-(a) = \varphi_1^-, \quad u_+(b) = \varphi_1^+ \\
u_-'(a) = \varphi_2^-, \quad u_+'(b) = \varphi_2^+ \\
u_-(\gamma) = u_+(\gamma) \\
u_-'(\gamma) = u_+'(\gamma) \\
k_+u_+''(\gamma) + k_+A_0u_+(\gamma) = k_-u_-''(\gamma) + k_-A_0u_-(\gamma) \\
k_+u_+^{(3)}(\gamma) + k_+A_0u_+'(\gamma) - l_+u_+'(\gamma) = k_-u_-^{(3)}(\gamma) + k_-A_0u_-'(\gamma) - l_-u_-'(\gamma),
\end{cases}
$$

where $f_- \in L^p(a,\gamma;L^p(\omega))$, $f_+ \in L^p(\gamma,b;L^p(\omega))$ and $p \in (1,+\infty)$, with $u(x) := u(x,\cdot)$ and $f(x) := f(x,\cdot)$.

Then, we will consider a generalization of this problem with $(-A, D(-A))$, instead of $(-A_0, D(-A_0))$, a BIP operator of angle $\theta \in (0,\pi)$ on a UMD space X, see Section 3 below for the definitions of BIP operator and UMD spaces, and $f \in L^p(a,b;X)$.

More precisely, we study the following transmission problem (P):

$$
\text{(P)}\begin{cases}
(EQ)\begin{cases}
u_-^{(4)}(x)+(2A-\dfrac{l_-}{k_-}I)u_-''(x)+(A^2-\dfrac{l_-}{k_-}A)u_-(x) &= f_-(x), \quad \text{for a.e. } x\in(a,\gamma)\\
u_+^{(4)}(x)+(2A-\dfrac{l_+}{k_+}I)u_+''(x)+(A^2-\dfrac{l_+}{k_+}A)u_+(x) &= f_+(x), \quad \text{for a.e. } x\in(\gamma,b)
\end{cases}\\
(BC)\begin{cases}
u_-(a) = \varphi_1^-, & u_+(b) = \varphi_1^+\\
u_-'(a) = \varphi_2^-, & u_+'(b) = \varphi_2^+
\end{cases}\\
(TC)\begin{cases}
u_-(\gamma) = u_+(\gamma)\\
u_-'(\gamma) = u_+'(\gamma)\\
k_+u_+^{(3)}(\gamma)+k_+Au_+'(\gamma)-l_+u_+'(\gamma) = k_-u_-^{(3)}(\gamma)+k_-Au_-'(\gamma)-l_-u_-'(\gamma)\\
k_+u_+''(\gamma)+k_+Au_+(\gamma) = k_-u_-''(\gamma)+k_-Au_-(\gamma).
\end{cases}
\end{cases}
$$

The transmission conditions (TC) will be divided into

$$
(TC1)\begin{cases}
u_-(\gamma) &= u_+(\gamma)\\
u_-'(\gamma) &= u_+'(\gamma),
\end{cases}
$$

and

$$
(TC2)\begin{cases}
k_+u_+^{(3)}(\gamma)+k_+Au_+'(\gamma)-l_+u_+'(\gamma) &= k_-u_-^{(3)}(\gamma)+k_-Au_-'(\gamma)-l_-u_-'(\gamma)\\
k_+u_+''(\gamma)+k_+Au_+(\gamma) &= k_-u_-''(\gamma)+k_-Au_-(\gamma).
\end{cases}
$$

Note that $(TC2)$ is well defined in virtue of Lemma 3.2, see Section 3.2 below.

We will search a classical solution of problem (P), that is a solution u such that

$$
\begin{cases}
u_- := u_{|(a,\gamma)} \in W^{4,p}(a,\gamma;X)\cap L^p(a,\gamma;D(A^2)), \quad u_-''\in L^p(a,\gamma;D(A)),\\
u_+ := u_{|(\gamma,b)} \in W^{4,p}(\gamma,b;X)\cap L^p(\gamma,b;D(A^2)), \quad u_+''\in L^p(\gamma,b;D(A)),
\end{cases}
\tag{2}
$$

and which satisfies $(EQ)-(BC)-(TC)$.

3 Assumptions, consequences and statement of results

3.1 The class $\mathbf{BIP}(\mathbf{X},\theta)$

In all the paper, $(X,\|\cdot\|)$ is a complex Banach space. Recall, see [12], p.19, that a closed linear operator T_1 is called sectorial of angle $\alpha\in(0,\pi)$ if

$$
\begin{array}{ll}
i) & \sigma(T_1)\subset\overline{S_\alpha},\\
ii) & \forall\,\alpha'\in(\alpha,\pi),\quad \sup\left\{\|\lambda(\lambda I-T_1)^{-1}\|_{\mathcal{L}(X)}:\ \lambda\in\mathbb{C}\setminus\overline{S_{\alpha'}}\right\}<\infty,
\end{array}
$$

where

$$
S_\alpha := \{z\in\mathbb{C}: z\neq 0 \text{ and } |\arg z|<\alpha\}. \tag{3}
$$

It is known that any injective sectorial operator T_1 admits imaginary powers T_1^{is}, $s\in\mathbb{R}$, but, in general, T_1^{is} is not bounded, see [13], p. 342. Let $\theta\in[0,\pi)$. We denote by $\text{BIP}(X,\theta)$, see [19], p.430, the class of sectorial injective operators T_1 such that

$$
\begin{array}{ll}
i) & \overline{D(T_1)}=\overline{R(T_1)}=X,\\
ii) & \forall\,s\in\mathbb{R},\quad T_1^{is}\in\mathcal{L}(X),\\
iii) & \exists\,C\geq 1,\ \forall\,s\in\mathbb{R},\quad ||T_1^{is}||_{\mathcal{L}(X)}\leq Ce^{|s|\theta}.
\end{array}
$$

In this case, $\overline{D(T_1) \cap R(T_1)} = X$, see [12], proof of Proposition 3.2.1, c), p. 71.

We will use the well-known Dore-Venni theorem, see [6] and its generalization in [19], which needs to consider a UMD space X. Recall that a Banach space X is a UMD space if and only if for some $p \in (1, +\infty)$ and thus for all p, the Hilbert transform is bounded from $L^p(\mathbb{R}, X)$ into itself.

3.2 Interpolation spaces

Here we recall some properties of real interpolation spaces in particular cases.
Let $T_2 : D(T_2) \subset X \longrightarrow X$ be a linear operator such that

$$(0, +\infty) \subset \rho(T_2) \quad \text{and} \quad \exists\, C > 0 : \forall\, t > 0, \quad \|t(T_2 - tI)^{-1}\|_{\mathcal{L}(X)} \leqslant C. \tag{4}$$

Then, for $\theta \in (0, 1)$ and $q \in [1, +\infty]$, we can define the real interpolation space

$$(D(T_2), X)_{\theta,q} := \left\{\psi \in X : t \longmapsto t^{1-\theta}\|T_2(T_2 - tI)^{-1}\psi\|_X \in L^q_*(0, +\infty)\right\},$$

see [11], p. 665, Teorema 3.

In [22], p. 78, this space is denoted by $(X, D(T_2))_{1-\theta,q}$. We set, for any $k \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned} (D(T_2), X)_{k+\theta,q} &:= \left\{\psi \in D(T_2^k) : T_2^k\psi \in (D(T_2), X)_{\theta,q}\right\}, \\ (X, D(T_2))_{k+\theta,q} &:= \left\{\psi \in D(T_2^k) : T_2^k\psi \in (X, D(T_2))_{\theta,q}\right\}. \end{aligned}$$

The general situation of the real interpolation space $(X_0, X_1)_{\theta,q}$ with X_0, X_1 two Banach spaces such that $X_0 \hookrightarrow X_1$, is described in [15].

Note that for an operator T_2 satisfying (4), T_2^k is closed for any $k \in \mathbb{N} \setminus \{0\}$ since $\rho(T_2) \neq \emptyset$; consequently we can consider $(D(T_2^k), X)_{\theta,q} = (X, D(T_2^k))_{1-\theta,q}$.

We have the two following lemmas.

Lemma 3.1. *Let $\theta, \theta' \in (0, 1)$, $k, n, m \in \mathbb{N} \setminus \{0\}$, $p \in [1, +\infty]$ and T_2 be a linear operator satisfying* (4).

i) If $\dfrac{k\theta}{n} \notin \mathbb{N}$, then $(X, D(T_2^k))_{\theta,p} = (X, D(T_2^n))_{\frac{k\theta}{n},p}$.

ii) If $n \leqslant k \leqslant m$, then

$$(D(T_2^k), D(T_2^n))_{\theta,p} = (D(T_2^m), X)_{\tau,p},$$

where τ satisfies $k(1 - \theta) + n\theta = m(1 - \tau)$.

iii) If $k\theta' < 1$, then $(D(T_2^k), X)_{\theta',p} \subset D(T_2^{k-1})$.

For statement *i*), see [16], (2.1.13), p. 43. For *ii*), see [11], p. 676, Teorema 6. For *iii*), we apply *ii*) with $n = k - 1$, $m = k$ and $\theta = k\theta' \in (0, 1)$, then

$$(D(T_2^k), D(T_2^{k-1}))_{k\theta',p} = (D(T_2^k), X)_{\theta',p},$$

which gives $(D(T_2^k), X)_{\theta',p} \subset D(T_2^{k-1})$. This inclusion can also be found by writing

$$\begin{aligned} (D(T_2^k), X)_{\theta',p} &= (X, D(T_2^k))_{1-\theta',p} &&= (X, D(T_2))_{k-k\theta',p} \\ &= (X, D(T_2))_{(k-1)+(1-k\theta'),p} &&= (D(T_2), X)_{(k-1)+k\theta',p} \subset D(T_2^{k-1}). \end{aligned}$$

Lemma 3.2. *Let T_2 be a linear operator satisfying* (4)*. Let u such that*

$$u \in W^{n,p}(a_1, b_1; X) \cap L^p(a_1, b_1; D(T_2^k)),$$

where $a_1, b_1 \in \mathbb{R}$ with $a_1 < b_1$, $n, k \in \mathbb{N} \setminus \{0\}$ and $p \in (1, +\infty)$. Then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $s \in \{a_1, b_1\}$, we have

$$u^{(j)}(s) \in (D(T_2^k), X)_{\frac{j}{n}+\frac{1}{np},p}.$$

This result is proved in [11], p. 678, Teorema 2'.

3.3 Hypotheses

In all the sequel, $k_+, k_-, l_+, l_- \in \mathbb{R}_+ \setminus \{0\}$, A denotes a closed linear operator in X and we set

$$r_+ = \frac{l_+}{k_+} \quad \text{and} \quad r_- = \frac{l_-}{k_-}.$$

We assume the following hypotheses:

(H_1) X is a UMD space,

(H_2) $0 \in \rho(A)$,

(H_3) $-A \in \text{BIP}(X, \theta_A)$ for some $\theta_A \in (0, \pi)$,

(H_4) $\sigma(A) \subset (-\infty, 0)$ and $\forall \theta \in (0, \pi)$, $\sup\limits_{\lambda \in S_\theta} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < +\infty$,

Note that (H_4) means that $-A$ is a sectorial operator of any angle $\theta \in (0, \pi)$. Let us give some consequences of our assumptions.

3.4 Consequences

1. Note that A_0 satisfies all the previous hypotheses with $X = L^q(\omega)$, $q \in (1, +\infty)$.

2. To solve each equation of (EQ) in the scalar case (with $-A > 0$), it is necessary to introduce the roots $\pm\sqrt{-A + r_\pm}$, $\pm\sqrt{-A}$ of the characteristic equations

$$x^4 + (2A - r_\pm)x^2 + (A^2 - r_\pm A) = 0,$$

this is why, in our operational case, we consider the operators

$$L_- := -\sqrt{-A + r_- I}, \quad L_+ := -\sqrt{-A + r_+ I} \quad \text{and} \quad M := -\sqrt{-A}. \tag{5}$$

Due to (H_3), $-A$, $-A + r_- I$ and $-A + r_+ I$ are sectorial operators, so the existence of L_-, L_+ and M is ensured, see for instance [12], e), p.25.

3. Applying Proposition 3.1.9, p. 65, in [12], we have $D(L_-) = D(L_+) = D(M)$. Thus, for $n, m \in \mathbb{N}$ and $m \leqslant n$

$$D(L_\pm^n) = D(M^n) = D(L_\pm^m M^{n-m}) = D(M^m L_\pm^{n-m}).$$

4. Due to (H_3), $-A + r_- I \in \text{BIP}(X, \theta_A)$ and $-A + r_+ I \in \text{BIP}(X, \theta_A)$, see [19], Theorem 3, p. 437, from which we deduce that

$$-L_-, -L_+, -M \in \text{BIP}(X, \theta_A/2),$$

see [12], Proposition 3.2.1, e), p. 71. Since $0 < \theta_A/2 < \pi/2$, L_-, L_+ and M generate bounded analytic semigroups $(e^{xL_-})_{x \geqslant 0}$, $(e^{xL_+})_{x \geqslant 0}$ and $(e^{xM})_{x \geqslant 0}$, see [19], Theorem 2, p. 437. Moreover, from [19], Theorem 4, p. 441, we get

$$-(L_- + M), -(L_+ + M) \in \text{BIP}(X, \theta_A/2 + \varepsilon),$$

for any $\varepsilon \in (0, \pi/2 - \theta_A/2)$. So from [19], Theorem 2, p. 437, $L_- + M$ and $L_+ + M$ generate bounded analytic semigroups $(e^{x(L_- + M)})_{x \geqslant 0}$ and $(e^{x(L_+ + M)})_{x \geqslant 0}$.

5. From (H_2) and (H_3), we deduce that $0 \in \rho(M) \cap \rho(L_-) \cap \rho(L_+)$. Thus, assumptions (H_1), (H_2) and (H_3) lead us to apply the Dore-Venni theorem, see [6], to obtain $0 \in \rho(L_+ + M)$ and $0 \in \rho(L_- + M)$.

6. It follows from (5) that

$$\forall\, \psi \in D(M^2), \quad (L_+^2 - M^2)\psi = r_+ \psi \quad \text{and} \quad (L_-^2 - M^2)\psi = r_- \psi. \tag{6}$$

and also

$$\forall\, \psi \in D(M), \;\; (L_+ - M)\psi = r_+(L_+ + M)^{-1}\psi \;\; \text{and} \;\; (L_- - M)\psi = r_-(L_- + M)^{-1}\psi. \tag{7}$$

3.5 The main results

To solve problem (P), we introduce two problems:

$$(P_-)\begin{cases} u_-^{(4)}(x)+(2A-r_-\,I)u_-''(x)+(A^2-r_-\,A)u_-(x)=f_-(x), & \text{for a.e. } x\in(a,\gamma)\\ u_-(a) \;=\; \varphi_1^-, \quad u_-(\gamma) \;=\; \psi_1\\ u_-'(a) \;=\; \varphi_2^-, \quad u_-'(\gamma) \;=\; \psi_2, \end{cases}$$

and

$$(P_+)\begin{cases} u_+^{(4)}(x)+(2A-r_+\,I)u_+''(x)+(A^2-r_+\,A)u_+(x)=f_+(x), & \text{for a.e. } x\in(\gamma,b)\\ u_+(\gamma) \;=\; \psi_1, \quad u_+(b) \;=\; \varphi_1^+\\ u_+'(\gamma) \;=\; \psi_2, \quad u_+'(b) \;=\; \varphi_2^+. \end{cases}$$

Remark 3.3. u is a classical solution of (P) if and only if there exist $\psi_1,\psi_2\in X$ such that

$$\begin{array}{ll} (i) & u_- \text{ is a classical solution of } (P_-),\\ (ii) & u_+ \text{ is a classical solution of } (P_+),\\ (iii) & u_- \text{ and } u_+ \text{ satisfy } (TC2). \end{array}$$

So our goal is to prove that there exists a unique couple (ψ_1,ψ_2) which satisfies (i), (ii) and (iii). This will lead us to obtain our main result.

Theorem 3.4. *Let $f_-\in L^p(a,\gamma;X)$ and $f_+\in L^p(\gamma,b;X)$. Assume that (H_1), (H_2), (H_3), (H_4) hold. Then, there exists a unique classical solution u, see definition* (2)*, of the transmission problem (P) if and only if*

$$\varphi_1^+,\varphi_1^-\in(D(A),X)_{1+\frac{1}{2p},p} \quad \textit{and} \quad \varphi_2^+,\varphi_2^-\in(D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{8}$$

Remark 3.5.

1. The proof of Theorem 3.4 uses operators L_-, L_+, M and also interpolation spaces $(D(M),X)_{3-j+\frac{1}{p},p}$, $j=0,1,2,3$. But from Lemma 3.1, we get

$$\begin{cases} (D(M),X)_{3+\frac{1}{p},p} = (D(A),X)_{1+\frac{1}{2p},p}, & (D(M),X)_{2+\frac{1}{p},p} = (D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p}\\ (D(M),X)_{1+\frac{1}{p},p} = (D(A),X)_{\frac{1}{2p},p}, & (D(M),X)_{\frac{1}{p},p} = (D(A),X)_{\frac{1}{2}+\frac{1}{2p},p}. \end{cases} \tag{9}$$

2. We can generalize this Theorem by considering a transmission problem between n habitats, with $n\in\mathbb{N}\setminus\{0\}$. It suffices to use Theorem 3.4 on the two first habitats and then apply it on the transmission problem between the second and the third habitat to solve the problem with $n=3$. By recurrence, we obtain the result.

As consequence of Theorem 3.4, we deduce some results for problem (P_{pde}) under some necessary boundary conditions. Let us consider the case $A=A_0$ (other cases can be treated).

Corollary 3.6. *Assume that ω is a bounded open set of $\mathbb{R}^{d-1}$ where $d\geq 2$ with C^2-boundary. Let $f_+\in L^p(\Omega_+)$ and $f_-\in L^p(\Omega_-)$ with $p\in(1,+\infty)$ and $p>d$; let $k_+,k_-,l_+,l_-\in\mathbb{R}_+\setminus\{0\}$. Then, there exists a unique solution u of (P_{pde}), such that*

$$u_-\in W^{4,p}(\Omega_-), \quad u_+\in W^{4,p}(\Omega_+),$$

if and only if

$$\varphi_1^\pm,\varphi_2^\pm\in W^{2,p}(\omega)\cap W_0^{1,p}(\omega), \quad \Delta\varphi_1^\pm,\in W^{2-\frac{1}{p},p}(\omega)\cap W_0^{1,p}(\omega) \quad \textit{and} \quad \Delta\varphi_2^\pm\in W^{1-\frac{1}{p},p}(\omega)\cap W_0^{1,p}(\omega).$$

Proof. Let $(x,y) \in (a,b)\times\omega$. Set $X := L^p(\omega)$. Using A_0 the linear operator defined by (1), we obtain that problem (P_{pde}) becomes problem (P). From [21], Proposition 3, p. 207, X satisfies (H_1) and from [9], Theorem 9.15 and Lemma 9.17, p. 241-242, A_0 satisfies (H_2). Moreover, (H_3) is satisfied from [20], Theorem C, p. 166-167. Moreover, since A_0 is the Laplace operator, from [12], Chapter 8, section 3, p. 232, (H_4) is satisfied. Finally, all the assumptions of Theorem 3.4 are satisfied. It follows that, there exists a unique classical solution of problem (P) if and only if (8) holds.

Now, it remains to show that if $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ satisfy (8), then the classical solution $u_\pm$ satisfies $u_\pm \in W^{4,p}(\Omega_\pm)$. To this end, we will make explicit the interpolation spaces that appear in (8). We have

$$(D(A_0), X)_{\frac{1}{2p},p} = \left(W^{2,p}(\omega)\cap W_0^{1,p}(\omega), L^p(\omega)\right)_{\frac{1}{2p},p},$$

and from [11], p. 683, proposizione 3 and p. 681, 1.10, and [22], p. 317, Theorem 1, since $2-\frac{1}{p} > 1$ is never integer, we have

$$\left(W^{2,p}(\omega), L^p(\omega)\right)_{\frac{1}{2p},p} = W^{2-\frac{1}{p},p}(\omega). \tag{10}$$

Set $\nu_1 := 2 - \dfrac{1}{p} - \dfrac{d-1}{p} = 2 - \dfrac{d}{p}$. Since $p > \dfrac{d}{2}$, we have $\nu_1 > 0$. From the Sobolev embedding theorem, see [22], section 4.6.1, p. 327-328, we have:

$$W^{2-\frac{1}{p},p}(\omega) \hookrightarrow C(\overline{\omega}).$$

Thus, the traces of the elements of the space described in (10) are well defined. From [10], Proposition 5.9, p. 334, and [22], section 4.3.3, Theorem, p. 321, we deduce that

$$(D(A_0), X)_{\frac{1}{2p},p} = \left\{\psi \in W^{2-\frac{1}{p},p}(\omega) : \psi = 0 \text{ on } \partial\omega\right\},$$

and

$$\begin{aligned}(D(A_0), X)_{1+\frac{1}{2p},p} &= \left\{\psi \in D(A_0) : A_0\psi \in W^{2-\frac{1}{p},p}(\omega) \ \text{ and } \ \Delta\psi = 0 \ \text{ on } \ \partial\omega\right\} \\ &= \left\{\psi \in W^{2,p}(\omega) : \Delta\psi \in W^{2-\frac{1}{p},p}(\omega) \ \text{ and } \ \psi = \Delta\psi = 0 \ \text{ on } \ \partial\omega\right\}.\end{aligned}$$

In the same way, we obtain

$$(D(A_0), X)_{\frac{1}{2}+\frac{1}{2p},p} = \left(W^{2,p}(\omega)\cap W_0^{1,p}(\omega), L^p(\omega)\right)_{\frac{1}{2}+\frac{1}{2p},p}. \tag{11}$$

Then, since $1-\frac{1}{p}$ is never an integer, from [11], Teorema 7, p. 681, we have

$$\left(W^{2,p}(\omega), L^p(\omega)\right)_{\frac{1}{2}+\frac{1}{2p},p} = B_{p,p}^{2(1-\frac{1}{2}-\frac{1}{2p})}(\omega) = B_{p,p}^{1-\frac{1}{p}}(\omega) = W^{1-\frac{1}{p},p}(\omega).$$

Set $\nu_2 := 1 - \dfrac{1}{p} - \dfrac{d-1}{p} = 1 - \dfrac{d}{p}$. Since $p > d$, we have $\nu_2 > 0$. From the Sobolev embedding theorem, see [22], section 4.6.1, p. 327-328, we have:

$$W^{1-\frac{1}{p},p}(\omega) \hookrightarrow C(\overline{\omega}).$$

Thus, the functions described in (11) are defined for any y in ω. From [10], Proposition 5.9, p. 334, and [22], section 4.3.3, Theorem, p. 321, we deduce that

$$(D(A_0), X)_{\frac{1}{2}+\frac{1}{2p},p} = \left\{\psi \in W^{1-\frac{1}{p},p}(\omega) : \psi = 0 \text{ on } \partial\omega\right\},$$

and

$$\begin{aligned}(D(A_0),X)_{1+\frac{1}{2}+\frac{1}{2p},p} &= \left\{\psi \in D(A_0) : A_0\psi \in (D(A_0),X)_{\frac{1}{2}+\frac{1}{2p},p}\right\} \\ &= \left\{\psi \in W^{2,p}(\omega) : \Delta\psi \in W^{1-\frac{1}{p},p}(\omega) \text{ and } \psi = \Delta\psi = 0 \text{ on } \partial\omega\right\}.\end{aligned}$$

We can apply on $u(x,\cdot)$, $x \in (a,b)$, in virtue of the Sobolev extension theorem on $\mathbb{R}^{d-1}$, an extension operator which maps continuously $L^p(\omega)$ into $L^p(\mathbb{R}^{d-1})$ and $W^{2,p}(\omega)$ into $W^{2,p}(\mathbb{R}^{d-1})$. Then, from the Mikhlin's Theorem, see [17], we deduce that

$$u_- \in W^{2,p}((a,\gamma)\times\omega) = W^{2,p}(\Omega_-) \quad \text{and} \quad u_+ \in W^{2,p}((\gamma,b)\times\omega) = W^{2,p}(\Omega_+).$$

By reiterating the same arguments to the other regularities, we obtain that $u_- \in W^{4,p}(\Omega_-)$ and $u_+ \in W^{4,p}(\Omega_+)$. □

Taking into account the result of Theorem 3.4, we can also obtain anisotropic result by considering $f_- \in L^p(a,\gamma;L^q(\omega))$ and $f_+ \in L^p(\gamma,b;L^q(\omega))$ with $p,q \in (1,+\infty)$.

4 Preliminary results

In all the sequel, we set

$$c = \gamma - a > 0 \quad \text{and} \quad d = b - \gamma > 0.$$

Applying Remark 3.3, we will solve problem (P) by studying first problems (P_-) and (P_+). To this end, we need the following invertibility result obtained in [14].

Lemma 4.1. *The operators U_+, U_-, V_+, $V_- \in \mathcal{L}(X)$ defined by*

$$\left\{\begin{array}{lll} U_+ &:=& I - e^{d(L_++M)} - r_+(L_++M)^2\left(e^{dM} - e^{dL_+}\right) \\ U_- &:=& I - e^{c(L_-+M)} - r_-(L_-+M)^2\left(e^{cM} - e^{cL_-}\right) \\ V_+ &:=& I - e^{d(L_++M)} + r_+(L_++M)^2\left(e^{dM} - e^{dL_+}\right) \\ V_- &:=& I - e^{c(L_-+M)} + r_-(L_-+M)^2\left(e^{cM} - e^{cL_-}\right), \end{array}\right. \tag{12}$$

are invertible with bounded inverse.

All these exponentials are well defined, see statement 4 of section 3.4. For a detailed proof, see [14], Proposition 5.4 with $k = r_-$ or $k = r_+$.

4.1 Problem $(\mathbf{P}_-)$

Proposition 4.2. *Let $f_- \in L^p(a,\gamma;X)$. Assume that (H_1), (H_2), (H_3), (H_4) hold. There exists a unique classical solution u_- of problem (P_-) if and only if*

$$\varphi_1^-, \psi_1 \in (D(A),X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_2^-, \psi_2 \in (D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{13}$$

Moreover

$$\begin{aligned} u_-(x) \;=\; & \left(e^{(x-a)M} - e^{(\gamma-x)M}\right)\alpha_1^- + \left(e^{(x-a)L_-} - e^{(\gamma-x)L_-}\right)\alpha_2^- \\ & + \left(e^{(x-a)M} + e^{(\gamma-x)M}\right)\alpha_3^- + \left(e^{(x-a)L_-} + e^{(\gamma-x)L_-}\right)\alpha_4^- + F_-(x), \end{aligned} \tag{14}$$

where

$$\begin{cases} \alpha_1^- & := & -\dfrac{1}{2r_-}(L_-+M)U_-^{-1}\left[L_-(I+e^{cL_-})\psi_1+(I-e^{cL_-})\psi_2+\tilde{\varphi}_1^-\right] \\ \alpha_2^- & := & \dfrac{1}{2r_-}(L_-+M)U_-^{-1}\left[M(I+e^{cM})\psi_1+(I-e^{cM})\psi_2+\tilde{\varphi}_2^-\right] \\ \alpha_3^- & := & \dfrac{1}{2r_-}(L_-+M)V_-^{-1}\left[L_-(I-e^{cL_-})\psi_1+(I+e^{cL_-})\psi_2+\tilde{\varphi}_3^-\right] \\ \alpha_4^- & := & -\dfrac{1}{2r_-}(L_-+M)V_-^{-1}\left[M(I-e^{cM})\psi_1+(I+e^{cM})\psi_2+\tilde{\varphi}_4^-\right], \end{cases} \tag{15}$$

$$\begin{cases} \tilde{\varphi_1}^- & := & -L_-\left(I+e^{cL_-}\right)\varphi_1^- - \left(I-e^{cL_-}\right)\left(F_-'(a)+F_-'(\gamma)-\varphi_2^-\right) \\ \tilde{\varphi_2}^- & := & -M\left(I+e^{cM}\right)\varphi_1^- - \left(I-e^{cM}\right)\left(F_-'(a)+F_-'(\gamma)-\varphi_2^-\right) \\ \tilde{\varphi_3}^- & := & L_-\left(I-e^{cL_-}\right)\varphi_1^- - \left(I+e^{cL_-}\right)\left(F_-'(\gamma)-F_-'(a)+\varphi_2^-\right) \\ \tilde{\varphi_4}^- & := & M\left(I-e^{cM}\right)\varphi_1^- - \left(I+e^{cM}\right)\left(F_-'(\gamma)-F_-'(a)+\varphi_2^-\right), \end{cases} \tag{16}$$

and F_- is the unique classical solution of problem

$$\begin{cases} u_-^{(4)}(x)+(2A-r_-\,I)u_-''(x)+(A^2-r_-\,A)u_-(x)=f_-(x), \quad \text{for a.e. } x\in(a,\gamma) \\ u_-(a)=u_-(\gamma)=u_-''(a)=u_-''(\gamma)=0. \end{cases} \tag{17}$$

Proof. From [14], Theorem 2.5, statement 2, there exists a unique classical solution u_- of (P_-) if and only if (13) holds. To obtain the representation formula (14)-(15)-(16) of u_-, we have adapted the representation formula (5.3)-(5.19)-(5.20) given in [14], where u, L, f, b, $F_{0,f}$, k, φ_1, φ_2, φ_3, φ_4 are respectively replaced by u_-, L_-, f_-, γ, F_-, r_-, φ_1^-, ψ_1, φ_2^-, ψ_2. □

Remark 4.3. In the previous proposition, due to (13), (15) and (16), we have

$$\alpha_i^- \in D(M), \quad \text{for } i=1,2,3,4.$$

Moreover, since F_- is a classical solution of (17), by Lemma 3.2, we deduce that, for $j=0,1,2,3$ and $s=a$ or γ

$$F_-^{(j)}(s)\in(D(M),X)_{3-j+\frac{1}{p},p}.$$

4.2 Problem $(\mathbf{P}_+)$

Proposition 4.4. *Let $f_+\in L^p(\gamma,b;X)$. Assume that (H_1), (H_2), (H_3), (H_4) hold. There exists a unique classical solution u_+ of (P_+) if and only if*

$$\varphi_1^+,\psi_1\in(D(A),X)_{1+\frac{1}{2p},p} \quad \textit{and} \quad \varphi_2^+,\psi_2\in(D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{18}$$

$$\begin{aligned} u_+(x) \quad = \quad & \left(e^{(x-\gamma)M}-e^{(b-x)M}\right)\alpha_1^+ + \left(e^{(x-\gamma)L_+}-e^{(b-x)L_+}\right)\alpha_2^+ \\ & + \left(e^{(x-\gamma)M}+e^{(b-x)M}\right)\alpha_3^+ + \left(e^{(x-\gamma)L_+}+e^{(b-x)L_+}\right)\alpha_4^+ + F_+(x), \end{aligned} \tag{19}$$

where

$$\begin{cases} \alpha_1^+ & = & \dfrac{1}{2r_+}(L_++M)U_+^{-1}\left[L_+(I+e^{dL_+})\psi_1-(I-e^{dL_+})\psi_2+\tilde{\varphi}_1^+\right] \\ \alpha_2^+ & = & -\dfrac{1}{2r_+}(L_++M)U_+^{-1}\left[M(I+e^{dM})\psi_1-(I-e^{dM})\psi_2+\tilde{\varphi}_2^+\right] \\ \alpha_3^+ & = & \dfrac{1}{2r_+}(L_++M)V_+^{-1}\left[L_+(I-e^{dL_+})\psi_1-(I+e^{dL_+})\psi_2+\tilde{\varphi}_3^+\right] \\ \alpha_4^+ & = & -\dfrac{1}{2r_+}(L_++M)V_+^{-1}\left[M(I-e^{dM})\psi_1-(I+e^{dM})\psi_2+\tilde{\varphi}_4^+\right], \end{cases} \tag{20}$$

$$\begin{cases} \tilde{\varphi_1}^+ &=& -L_+ \left(I + e^{dL_+}\right) \varphi_1^+ + \left(I - e^{dL_+}\right)\left(F'_+(b) + F'_+(\gamma) - \varphi_2^+\right) \\ \tilde{\varphi_2}^+ &=& -M \left(I + e^{dM}\right) \varphi_1^+ + \left(I - e^{dM}\right)\left(F'_+(b) + F'_+(\gamma) - \varphi_2^+\right) \\ \tilde{\varphi_3}^+ &=& L_+ \left(I - e^{dL_+}\right) \varphi_1^+ - \left(I + e^{dL_+}\right)\left(F'_+(b) - F'_+(\gamma) - \varphi_2^+\right) \\ \tilde{\varphi_4}^+ &=& M \left(I - e^{dM}\right) \varphi_1^+ - \left(I + e^{dM}\right)\left(F'_+(b) - F'_+(\gamma) - \varphi_2^+\right), \end{cases} \tag{21}$$

and F_+ is the unique classical solution of problem

$$\begin{cases} u_+^{(4)}(x) + (2A - r_+ I)u''_+(x) + (A^2 - r_+ A)u_+(x) = f_+(x), \quad \text{for a.e. } x \in (\gamma, b) \\ u_+(\gamma) = u_+(b) = u''_+(\gamma) = u''_+(b) = 0. \end{cases} \tag{22}$$

Proof. From [14], Theorem 2.5, statement 2, there exists a unique classical solution u_+ of (P_+) if and only if (18) holds. To obtain the representation formula (19)-(20)-(21) of u_+, we have adapted the representation formula (5.3)-(5.19)-(5.20) given in [14], where u, L, f, a, $F_{0,f}$, k, φ_1, φ_2, φ_3, φ_4 are respectively replaced by u_+, L_+, f_+, γ, F_+, r_+, ψ_1, φ_1^+, ψ_2, φ_2^+. □

Remark 4.5. In the previous proposition, due to (18), (20) and (21), we have

$$\alpha_i^+ \in D(M), \quad \text{for } i = 1, 2, 3, 4.$$

Moreover, since F_+ is a classical solution of (22), by Lemma 3.2, we deduce that, for $j = 0, 1, 2, 3$ and $s = \gamma$ or b

$$F_+^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{p},p}.$$

4.3 The transmission system

This section is devoted to the proof of Theorem 4.6 stated below, which gives the link between problem (P) and the following system

$$\begin{cases} \left(P_1^+ + P_1^-\right)\psi_1 &+& \left(P_2^- - P_2^+\right)\psi_2 &=& S_1 \\ M\left(P_2^- - P_2^+\right)\psi_1 &+& \left(P_3^+ + P_3^-\right)\psi_2 &=& S_2. \end{cases} \tag{23}$$

The coefficients are given by

$$\begin{cases} P_1^+ &=& k_+(L_+ + M)L_+\left(U_+^{-1}(I + e^{dM})(I + e^{dL_+}) + V_+^{-1}(I - e^{dM})(I - e^{dL_+})\right) \\ P_2^+ &=& k_+(L_+ + M)\left(U_+^{-1}(I + e^{dM})(I - e^{dL_+}) + V_+^{-1}(I - e^{dM})(I + e^{dL_+})\right) \\ P_3^+ &=& k_+(L_+ + M)\left(U_+^{-1}(I - e^{dM})(I - e^{dL_+}) + V_+^{-1}(I + e^{dM})(I + e^{dL_+})\right), \end{cases} \tag{24}$$

and similarly

$$\begin{cases} P_1^- &=& k_-(L_- + M)L_-\left(U_-^{-1}(I + e^{cM})(I + e^{cL_-}) + V_-^{-1}(I - e^{cM})(I - e^{cL_-})\right) \\ P_2^- &=& k_-(L_- + M)\left(U_-^{-1}(I + e^{cM})(I - e^{cL_-}) + V_-^{-1}(I - e^{cM})(I + e^{cL_-})\right) \\ P_3^- &=& k_-(L_- + M)\left(U_-^{-1}(I - e^{cM})(I - e^{cL_-}) + V_-^{-1}(I + e^{cM})(I + e^{cL_-})\right). \end{cases} \tag{25}$$

The second members are

$$\begin{aligned} S_1 &= -k_+(L_+ + M)\left(U_+^{-1}(I + e^{dM})\tilde{\varphi_1}^+ + V_+^{-1}(I - e^{dM})\tilde{\varphi_3}^+\right) \\ &\quad -k_-(L_- + M)\left(U_-^{-1}(I + e^{cM})\tilde{\varphi_1}^- + V_-^{-1}(I - e^{cM})\tilde{\varphi_3}^-\right) - 2M^{-1}R_1, \end{aligned} \tag{26}$$

with R_1 given by

$$R_1 = -k_+ F_+'''(\gamma) + k_+ M^2 F_+'(\gamma) + l_+ F_+'(\gamma) + k_- F_-'''(\gamma) - k_- M^2 F_-'(\gamma) - l_- F_-'(\gamma), \tag{27}$$

and

$$\begin{aligned} S_2 &= k_+(L_+ + M)\left(U_+^{-1}(I - e^{dL_+})\tilde{\varphi_2}^+ + V_+^{-1}(I + e^{dL_+})\tilde{\varphi_4}^+\right) \\ &\quad -k_-(L_- + M)\left(U_-^{-1}(I - e^{cL_-})\tilde{\varphi_2}^- + V_-^{-1}(I + e^{cL_-})\tilde{\varphi_4}^-\right). \end{aligned} \tag{28}$$

Theorem 4.6. *Let $f_- \in L^p(a,\gamma;X)$ and $f_+ \in L^p(\gamma,b;X)$. Assume that (H_1), (H_2), (H_3), (H_4) hold. Then, the transmission problem (P) has a unique classical solution if and only if the data $\varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^-$ satisfy* (8) *and system* (23) *has a unique solution (ψ_1, ψ_2) such that*

$$(\psi_1, \psi_2) \in (D(A), X)_{1+\frac{1}{2p},p} \times (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{29}$$

Proof. First, we assume that (P) admits a unique classical solution u. Setting

$$\psi_1 = u_-(\gamma) = u_+(\gamma) \quad \text{and} \quad \psi_2 = u_-'(\gamma) = u_+'(\gamma),$$

we get that u_- (respectively u_+) is the classical solution of (P_-) (respectively (P_+)). So, applying Proposition 4.2 (respectively Proposition 4.4), we obtain (8) and also (29). It remains to prove that (ψ_1, ψ_2) satisfies (23). To this end we use $(TC2)$ satisfied by u, that is

$$\begin{cases} k_+\left(u_+^{(3)}(\gamma) - M^2 u_+'(\gamma)\right) - l_+ u_+'(\gamma) - k_-\left(u_-^{(3)}(\gamma) - M^2 u_-'(\gamma)\right) + l_- u_-'(\gamma) = 0 \\ k_+\left(u_+''(\gamma) - M^2 u_+(\gamma)\right) - k_-\left(u_-''(\gamma) - M^2 u_-(\gamma)\right) = 0. \end{cases}$$

To make explicit this system, we use the expression of u_+ given in (19). It follows, for $x \in (\gamma, b)$

$$\begin{aligned} u_+(x) &= \left(e^{(x-\gamma)M} - e^{(b-x)M}\right)\alpha_1^+ + \left(e^{(x-\gamma)L_+} - e^{(b-x)L_+}\right)\alpha_2^+ \\ &\quad + \left(e^{(x-\gamma)M} + e^{(b-x)M}\right)\alpha_3^+ + \left(e^{(x-\gamma)L_+} + e^{(b-x)L_+}\right)\alpha_4^+ + F_+(x), \end{aligned}$$

$$\begin{aligned} u_+'(x) &= M\left(e^{(x-\gamma)M} + e^{(b-x)M}\right)\alpha_1^+ + L_+\left(e^{(x-\gamma)L_+} + e^{(b-x)L_+}\right)\alpha_2^+ \\ &\quad + M\left(e^{(x-\gamma)M} - e^{(b-x)M}\right)\alpha_3^+ + L_+\left(e^{(x-\gamma)L_+} - e^{(b-x)L_+}\right)\alpha_4^+ + F_+'(x), \end{aligned}$$

$$\begin{aligned} u_+''(x) &= M^2\left(e^{(x-\gamma)M} - e^{(b-x)M}\right)\alpha_1^+ + L_+^2\left(e^{(x-\gamma)L_+} - e^{(b-x)L_+}\right)\alpha_2^+ \\ &\quad + M^2\left(e^{(x-\gamma)M} + e^{(b-x)M}\right)\alpha_3^+ + L_+^2\left(e^{(x-\gamma)L_+} + e^{(b-x)L_+}\right)\alpha_4^+ + F_+''(x), \end{aligned}$$

$$\begin{aligned} u_+^{(3)}(x) &= M^3\left(e^{(x-\gamma)M} + e^{(b-x)M}\right)\alpha_1^+ + L_+^3\left(e^{(x-\gamma)L_+} + e^{(b-x)L_+}\right)\alpha_2^+ \\ &\quad + M^3\left(e^{(x-\gamma)M} - e^{(b-x)M}\right)\alpha_3^+ + L_+^3\left(e^{(x-\gamma)L_+} - e^{(b-x)L_+}\right)\alpha_4^+ + F_+'''(x). \end{aligned}$$

Then, in virtue of Lemma 3.2, we have

$$\begin{aligned} M^{-2}\left(u_+^{(3)}(\gamma) - M^2 u_+'(\gamma)\right) &= L_+(L_+^2 - M^2)M^{-2}\left(I + e^{dL_+}\right)\alpha_2^+ \\ &\quad + L_+(L_+^2 - M^2)M^{-2}\left(I - e^{dL_+}\right)\alpha_4^+ \\ &\quad + M^{-2}F_+'''(\gamma) - F_+'(\gamma). \end{aligned}$$

Furthermore, from (6), we obtain

$$\begin{aligned} M^{-2}\left(u_+^{(3)}(\gamma) - M^2 u_+'(\gamma)\right) &= \frac{l_+}{k_+}L_+ M^{-2}\left(I + e^{dL_+}\right)\alpha_2^+ \\ &\quad + \frac{l_+}{k_+}L_+ M^{-2}\left(I - e^{dL_+}\right)\alpha_4^+ + M^{-2}F_+'''(\gamma) - F_+'(\gamma), \end{aligned}$$

hence

$$u_+^{(3)}(\gamma) - M^2 u_+'(\gamma) = \frac{l_+}{k_+} L_+ \left(I + e^{dL_+}\right) \alpha_2^+ + \frac{l_+}{k_+} L_+ \left(I - e^{dL_+}\right) \alpha_4^+ + F_+'''(\gamma) - M^2 F_+'(\gamma),$$

it follows that

$$\begin{aligned} k_+ \left(u_+^{(3)}(\gamma) - M^2 u_+'(\gamma)\right) - l_+ u_+'(\gamma) &= -l_+ M \left(I + e^{dM}\right) \alpha_1^+ - l_+ M \left(I - e^{dM}\right) \alpha_3^+ \\ &\quad + k_+ F_+'''(\gamma) - k_+ M^2 F_+'(\gamma) - l_+ F_+'(\gamma). \end{aligned}$$

Note that, from Remark 4.5 and Lemma 3.2, all the terms in the previous equalities are justified. By the same arguments and using again (6) and also (22), we have

$$\begin{aligned} M^{-1}\left(u_+''(\gamma) - M^2 u_+(\gamma)\right) &= (L_+^2 - M^2) M^{-1} \left(I - e^{dL_+}\right) \alpha_2^+ + (L_+^2 - M^2) M^{-1} \left(I + e^{dL_+}\right) \alpha_4^+ \\ &= \frac{l_+}{k_+} M^{-1} \left(I - e^{dL_+}\right) \alpha_2^+ + \frac{l_+}{k_+} M^{-1} \left(I + e^{dL_+}\right) \alpha_4^+. \end{aligned}$$

Then, we obtain

$$k_+ \left(u_+''(\gamma) - M^2 u_+(\gamma)\right) = l_+ \left(I - e^{dL_+}\right) \alpha_2^+ + l_+ \left(I + e^{dL_+}\right) \alpha_4^+. \tag{30}$$

As previously, for u_-, we use (14) and get, for $x \in (a, \gamma)$

$$\begin{aligned} u_-(x) &= \left(e^{(x-a)M} - e^{(\gamma-x)M}\right) \alpha_1^- + \left(e^{(x-a)L_-} - e^{(\gamma-x)L_-}\right) \alpha_2^- \\ &+ \left(e^{(x-a)M} + e^{(\gamma-x)M}\right) \alpha_3^- + \left(e^{(x-a)L_-} + e^{(\gamma-x)L_-}\right) \alpha_4^- + F_-(x), \end{aligned}$$

$$\begin{aligned} u_-'(x) &= M\left(e^{(x-a)M} + e^{(\gamma-x)M}\right) \alpha_1^- + L_-\left(e^{(x-a)L_-} + e^{(\gamma-x)L_-}\right) \alpha_2^- \\ &+ M\left(e^{(x-a)M} - e^{(\gamma-x)M}\right) \alpha_3^- + L_-\left(e^{(x-a)L_-} - e^{(\gamma-x)L_-}\right) \alpha_4^- + F_-'(x), \end{aligned}$$

$$\begin{aligned} u_-''(x) &= M^2\left(e^{(x-a)M} - e^{(\gamma-x)M}\right) \alpha_1^- + L_-^2\left(e^{(x-a)L_-} - e^{(\gamma-x)L_-}\right) \alpha_2^- \\ &+ M^2\left(e^{(x-a)M} + e^{(\gamma-x)M}\right) \alpha_3^- + L_-^2\left(e^{(x-a)L_-} + e^{(\gamma-x)L_-}\right) \alpha_4^- + F_-''(x), \end{aligned}$$

$$\begin{aligned} u_-^{(3)}(x) &= M^3\left(e^{(x-a)M} + e^{(\gamma-x)M}\right) \alpha_1^- + L_-^3\left(e^{(x-a)L_-} + e^{(\gamma-x)L_-}\right) \alpha_2^- \\ &+ M^3\left(e^{(x-a)M} - e^{(\gamma-x)M}\right) \alpha_3^- + L_-^3\left(e^{(x-a)L_-} - e^{(\gamma-x)L_-}\right) \alpha_4^- + F_-'''(x). \end{aligned}$$

Then, in virtue of Lemma 3.2, we have

$$\begin{aligned} u_-^{(3)}(\gamma) - M^2 u_-'(\gamma) &= L_-(L_-^2 - M^2)\left(e^{cL_-} + I\right) \alpha_2^- + L_-(L^2 - M^2)\left(e^{cL_-} - I\right) \alpha_4^- \\ &\quad + F_-'''(\gamma) - M^2 F_-'(\gamma), \end{aligned}$$

hence, due to (6), we have

$$\begin{aligned} M^{-2}\left(u_-^{(3)}(\gamma) - M^2 u_-'(\gamma)\right) &= L_-(L_-^2 - M^2) M^{-2}\left(e^{cL_-} + I\right) \alpha_2^- + M^{-2} F_-'''(\gamma) \\ &\quad + L_-(L^2 - M^2) M^{-2}\left(e^{cL_-} - I\right) \alpha_4^- - F_-'(\gamma) \\ &= L_- \frac{l_-}{k_-} M^{-2}\left(e^{cL_-} + I\right) \alpha_2^- + M^{-2} F_-'''(\gamma) \\ &\quad + L_- \frac{l_-}{k_-} M^{-2}\left(e^{cL_-} - I\right) \alpha_4^- - F_-'(\gamma). \end{aligned}$$

Then, we obtain

$$\begin{aligned}
k_-\left(u_-^{(3)}(\gamma)-M^2u_-'(\gamma)\right)-l_-u_-'(\gamma) &= l_-L_-\left(e^{cL_-}+I\right)\alpha_2^-+l_-L_-\left(e^{cL_-}-I\right)\alpha_4^- \\
&\quad -l_-M\left(e^{cM}+I\right)\alpha_1^--l_-L_-\left(e^{cL_-}+I\right)\alpha_2^- \\
&\quad -l_-M\left(e^{cM}-I\right)\alpha_3^--l_-L_-\left(e^{cL_-}-I\right)\alpha_4^- \\
&\quad +k_-F_-'''(\gamma)-k_-M^2F_-'(\gamma)-l_-F_-'(\gamma) \\
&= -l_-M\left(e^{cM}+I\right)\alpha_1^--l_-M\left(e^{cM}-I\right)\alpha_3^- \\
&\quad +k_-F_-'''(\gamma)-k_-M^2F_-'(\gamma)-l_-F_-'(\gamma).
\end{aligned}$$

Furthermore, from (6), we have

$$\begin{aligned}
M^{-1}\left(u_-''(\gamma)-M^2u_-(\gamma)\right) &= (L_-^2-M^2)M^{-1}\left(e^{cL_-}-I\right)\alpha_2^-+(L_-^2-M^2)M^{-1}\left(I+e^{cL_-}\right)\alpha_4^- \\
&= \frac{l_-}{k_-}M^{-1}\left(e^{cL_-}-I\right)\alpha_2^-+\frac{l_-}{k_-}M^{-1}\left(I+e^{cL_-}\right)\alpha_4^-.
\end{aligned}$$

Then, we deduce the following equality:

$$k_-\left(u_-''(\gamma)-M^2u_-(\gamma)\right)=l_-\left(e^{cL_-}-I\right)\alpha_2^-+l_-\left(I+e^{cL_-}\right)\alpha_4^-. \tag{31}$$

Note that, from Remark 4.3 and Lemma 3.2, all the terms in the previous equalities are justified. It follows, from (30) and (31), that system $(TC2)$ becomes

$$\begin{cases}
-l_+M\left(I+e^{dM}\right)\alpha_1^+-l_+M\left(I-e^{dM}\right)\alpha_3^+=-l_-M\left(I+e^{cM}\right)\alpha_1^-+l_-M\left(I-e^{cM}\right)\alpha_3^-+R_1 \\[2mm]
\quad l_+\left(I-e^{dL_+}\right)\alpha_2^++\quad l_+\left(I+e^{dL_+}\right)\alpha_4^+=\quad -l_-\left(I-e^{cL_-}\right)\alpha_2^-+\ l_-\left(I+e^{cL_-}\right)\alpha_4^-,
\end{cases}$$

where, R_1 is given by (27). Thus

$$\begin{cases}
-l_+\left((\alpha_1^++\alpha_3^+)-e^{dM}(\alpha_3^+-\alpha_1^+)\right)=l_-\left((\alpha_3^--\alpha_1^-)-e^{cM}(\alpha_1^-+\alpha_3^-)\right)+M^{-1}R_1 \\[2mm]
\ l_+\left((\alpha_2^++\alpha_4^+)+e^{dL_+}(\alpha_4^+-\alpha_2^+)\right)=l_-\left((\alpha_4^--\alpha_2^-)+e^{cL_-}(\alpha_2^-+\alpha_4^-)\right),
\end{cases} \tag{32}$$

But, from (20), (21), (15) and (16), we have

$$\begin{aligned}
\alpha_1^++\alpha_3^+ &= \frac{1}{2r_+}(L_++M)U_+^{-1}\left(L_+(I+e^{dL_+})\psi_1-(I-e^{dL_+})\psi_2+\tilde{\varphi_1}^+\right) \\
&\quad +\frac{1}{2r_+}(L_++M)V_+^{-1}\left(L_+(I-e^{dL_+})\psi_1-(I+e^{dL_+})\psi_2+\tilde{\varphi_3}^+\right),
\end{aligned}$$

$$\begin{aligned}
\alpha_3^+-\alpha_1^+ &= -\frac{1}{2r_+}(L_++M)U_+^{-1}\left(L_+(I+e^{dL_+})\psi_1-(I-e^{dL_+})\psi_2+\tilde{\varphi_1}^+\right) \\
&\quad +\frac{1}{2r_+}(L_++M)V_+^{-1}\left(L_+(I-e^{dL_+})\psi_1-(I+e^{dL_+})\psi_2+\tilde{\varphi_3}^+\right),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1^-+\alpha_3^- &= -\frac{1}{2r_-}(L_-+M)U_-^{-1}\left(L_-(I+e^{cL_-})\psi_1+(I-e^{cL_-})\psi_2+\tilde{\varphi_1}^-\right) \\
&\quad +\frac{1}{2r_-}(L_-+M)V_-^{-1}\left(L_-(I-e^{cL_-})\psi_1+(I+e^{cL_-})\psi_2+\tilde{\varphi_3}^-\right),
\end{aligned}$$

$$\begin{aligned}
\alpha_3^--\alpha_1^- &= \frac{1}{2r_-}(L_-+M)U_-^{-1}\left(L_-(I+e^{cL_-})\psi_1+(I-e^{cL_-})\psi_2+\tilde{\varphi_1}^-\right) \\
&\quad +\frac{1}{2r_-}(L_-+M)V_-^{-1}\left(L_-(I-e^{cL_-})\psi_1+(I+e^{cL_-})\psi_2+\tilde{\varphi_3}^-\right).
\end{aligned}$$

So, using (24) and (25), the first line of system (32) writes

$$\left(P_1^+ + P_1^-\right)\psi_1 + \left(P_2^- - P_2^+\right)\psi_2 = S_1, \tag{33}$$

where S_1 is given by (26). By the same way, we have

$$\begin{array}{rcl}
\alpha_2^+ + \alpha_4^+ & = & -\dfrac{1}{2r_+}(L_+ + M)U_+^{-1}\left(M(I + e^{dM})\psi_1 - (I - e^{dM})\psi_2 + \tilde{\varphi_2}^+\right) \\
& & -\dfrac{1}{2r_+}(L_+ + M)V_+^{-1}\left(M(I - e^{dM})\psi_1 - (I + e^{dM})\psi_2 + \tilde{\varphi_4}^+\right), \\[2ex]
\alpha_4^+ - \alpha_2^+ & = & \dfrac{1}{2r_+}(L_+ + M)U_+^{-1}\left(M(I + e^{dM})\psi_1 - (I - e^{dM})\psi_2 + \tilde{\varphi_2}^+\right) \\
& & -\dfrac{1}{2r_+}(L_+ + M)V_+^{-1}\left(M(I - e^{dM})\psi_1 - (I + e^{dM})\psi_2 + \tilde{\varphi_4}^+\right), \\[2ex]
\alpha_2^- + \alpha_4^- & = & \dfrac{1}{2r_-}(L_- + M)U_-^{-1}\left(M(I + e^{cM})\psi_1 + (I - e^{cM})\psi_2 + \tilde{\varphi_2}^-\right) \\
& & -\dfrac{1}{2r_-}(L_- + M)V_-^{-1}\left(M(I - e^{cM})\psi_1 + (I + e^{cM})\psi_2 + \tilde{\varphi_4}^-\right), \\[2ex]
\alpha_4^- - \alpha_2^- & = & -\dfrac{1}{2r_-}(L_- + M)U_-^{-1}\left(M(I + e^{cM})\psi_1 + (I - e^{cM})\psi_2 + \tilde{\varphi_2}^-\right) \\
& & -\dfrac{1}{2r_-}(L_- + M)V_-^{-1}\left(M(I - e^{cM})\psi_1 + (I + e^{cM})\psi_2 + \tilde{\varphi_4}^-\right).
\end{array}$$

So, using (24) and (25), the second line of system (32) writes

$$M\left(P_2^- - P_2^+\right)\psi_1 + \left(P_3^+ + P_3^-\right)\psi_2 = S_2, \tag{34}$$

where S_2 is given by (28). Then, due to (33) and (34), (ψ_1, ψ_2) is the expected solution of (23).

Conversely, if (8) holds and system (23) has a unique solution (ψ_1, ψ_2) which satisfies (29), then considering u_- (respectively u_+) the unique classical solution of (P_-) (respectively (P_+)), we get that u is the unique classical solution of (P). □

4.4 Functional calculus

Due to Theorem 4.6, to prove Theorem 3.4, it remains to solve system (23). This will be done by using functional calculus to rewrite the operators defined in (12), (24) and (25) and to inverse the determinant operator of system (23).

To this end, we first recall some classical notations. For $\theta \in (0, \pi)$, we denote by $H(S_\theta)$ the space of holomorphic functions on S_θ (defined by (3)) with values in $\mathbb{C}$. Moreover, we consider the following subspace of $H(S_\theta)$:

$$\mathcal{E}_\infty(S_\theta) := \left\{f \in H(S_\theta) : f = O(|z|^{-s}) \ (|z| \to +\infty) \text{ for some } s > 0\right\}.$$

In other words, $\mathcal{E}_\infty(S_\theta)$ is the space of polynomial decreasing holomorphic functions at $+\infty$. Let T be an invertible sectorial operator of angle $\theta_T \in (0, \pi)$. If $f \in \mathcal{E}_\infty(S_\theta)$, with $\theta \in (\theta_T, \pi)$, then we can define, by functional calculus, $f(T) \in \mathcal{L}(X)$, see [12], p. 45.

Then, we recall a result from [14], section 5.2, Lemma 5.3.

Lemma 4.7. *Let P be an invertible sectorial operator in X with angle θ, for all $\theta \in (0, \pi)$. Let $G \in H(S_\theta)$, for some $\theta \in (0, \pi)$, such that*

(*i*) $1 - G \in \mathcal{E}_\infty(S_\theta)$, *(ii)* $G(x) \neq 0$ *for any* $x \in \mathbb{R}_+ \setminus \{0\}$.

Then, $G(P) \in \mathcal{L}(X)$*, is invertible with bounded inverse.*

Now, in order to apply the previous lemma to inverse the determinant of system (23), we introduce some holomorphic functions and study them on the positive real axis.

Let $r, \delta > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}_-$. We set

$$\left\{\begin{array}{rcl} u_{\delta,r}(z) & = & 1 - e^{-\delta(\sqrt{z+r}+\sqrt{z})} - \dfrac{1}{r}(\sqrt{z+r}+\sqrt{z})^2\left(e^{-\delta\sqrt{z}} - e^{-\delta\sqrt{z+r}}\right) \\ v_{\delta,r}(z) & = & 1 - e^{-\delta(\sqrt{z+r}+\sqrt{z})} + \dfrac{1}{r}(\sqrt{z+r}+\sqrt{z})^2\left(e^{-\delta\sqrt{z}} - e^{-\delta\sqrt{z+r}}\right), \end{array}\right.$$

and when $u_{\delta,r}(z) \neq 0$, $v_{\delta,r}(z) \neq 0$

$$\left\{\begin{array}{rcl} f_{\delta,r,1}(z) & = & \left(\sqrt{z+r}+\sqrt{z}\right)\sqrt{z+r}\,u_{\delta,r}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)\left(1+e^{-\delta\sqrt{z+r}}\right) \\ & & +\left(\sqrt{z+r}+\sqrt{z}\right)\sqrt{z+r}\,v_{d,r}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)\left(1-e^{-\delta\sqrt{z+r}}\right) \\ f_{\delta,r,2}(z) & = & -\left(\sqrt{z+r}+\sqrt{z}\right)u_{\delta,r}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)\left(1-e^{-\delta\sqrt{z+r}}\right) \\ & & -\left(\sqrt{z+r}+\sqrt{z}\right)v_{\delta,r}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)\left(1+e^{-\delta\sqrt{z+r}}\right) \\ f_{\delta,r,3}(z) & = & -\left(\sqrt{z+r}+\sqrt{z}\right)u_{\delta,r}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)\left(1-e^{-\delta\sqrt{z+r}}\right) \\ & & -\left(\sqrt{z+r}+\sqrt{z}\right)v_{\delta,r}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)\left(1+e^{-\delta\sqrt{z+r}}\right). \end{array}\right. \tag{35}$$

Remark 4.8. Let $r, \delta, x > 0$. From [14], Lemma 5.2, section 5.2, p. 369, we have $u_{\delta,r}(x) > 0$ and $v_{\delta,r}(x) > 0$. Then, we obtain

$$f_{\delta,r,1}(x) > 0 \quad \text{and} \quad f_{\delta,r,2}(x), f_{\delta,r,3}(x) < 0.$$

Moreover, for $z \in \mathbb{C} \setminus \mathbb{R}_-$, we define

$$\begin{array}{rcl} g_{\delta,r}(z) & = & -\sqrt{z+r}\left((1-e^{-2\delta(\sqrt{z+r}+\sqrt{z})})^2 - \dfrac{1}{r^2}(\sqrt{z+r}+\sqrt{z})^4\left(e^{-2\delta\sqrt{z}} - e^{-2\delta\sqrt{z+r}}\right)^2\right) \\ & & +\sqrt{z}\left((1-e^{-\delta(\sqrt{z+r}+\sqrt{z})})^2 + \dfrac{1}{r}(\sqrt{z+r}+\sqrt{z})^2\left(e^{-\delta\sqrt{z}} - e^{-\delta\sqrt{z+r}}\right)^2\right)^2, \end{array}$$

and also, when $u_{\delta,r}(z) \neq 0$, $v_{\delta,r}(z) \neq 0$

$$h_{\delta,r}(z) = 4(\sqrt{z+r}+\sqrt{z})^2 u_{\delta,r}^{-2}(z) v_{\delta,r}^{-2}(z) g_{\delta,r}(z).$$

Lemma 4.9. *Let* $r, \delta, x > 0$*. We have*

$$g_{\delta,r}(x) < 0 \quad \textit{and} \quad h_{\delta,r}(x) < 0.$$

Proof. Set

$$s = \sqrt{x} \quad \text{and} \quad t = \sqrt{x+r}.$$

Then, we have

$$
\begin{aligned}
g_{\delta,r}(x) &= -t\left((1-e^{-2\delta(s+t)})^2 - \frac{1}{r^2}(s+t)^4\left(e^{-2\delta s}-e^{-2\delta t}\right)^2\right)\\
&\quad +s\left((1-e^{-\delta(s+t)})^2 + \frac{1}{r}(s+t)^2\left(e^{-\delta s}-e^{-\delta t}\right)^2\right)^2\\
&= -t\left((1-e^{-\delta(s+t)})^2(1+e^{-\delta(s+t)})^2 - \frac{1}{r^2}(s+t)^4\left(e^{-\delta s}-e^{-\delta t}\right)^2\left(e^{-\delta s}+e^{-\delta t}\right)^2\right)\\
&\quad +s\left((1-e^{-\delta(s+t)})^4 + 2\frac{1}{r}(s+t)^2(1-e^{-\delta(s+t)})^2\left(e^{-\delta s}-e^{-\delta t}\right)^2\right)\\
&\quad +s\frac{1}{r^2}(s+t)^4\left(e^{-\delta s}-e^{-\delta t}\right)^4\\
&= \frac{s(s+t)^4}{r^2}\left(e^{-\delta s}-e^{-\delta t}\right)^4 + \frac{t(s+t)^4}{r^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2\left(e^{-\delta s}+e^{-\delta t}\right)^2\\
&\quad +s(1-e^{-\delta(s+t)})^4 - t(1-e^{-\delta(s+t)})^2(1+e^{-\delta(s+t)})^2\\
&\quad +2\frac{s(s+t)^2}{r}(1-e^{-\delta(s+t)})^2\left(e^{-\delta s}-e^{-\delta t}\right)^2\\
&= \frac{(s+t)^4}{r^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2\left(s\left(e^{-\delta s}-e^{-\delta t}\right)^2 + t\left(e^{-\delta s}+e^{-\delta t}\right)^2\right)\\
&\quad +(1-e^{-\delta(s+t)})^2\left(s(1-e^{-\delta(s+t)})^2 - t(1+e^{-\delta(s+t)})^2\right)\\
&\quad +2\frac{s(s+t)^2}{r}(1-e^{-\delta(s+t)})^2\left(e^{-\delta s}-e^{-\delta t}\right)^2\\
&= \frac{(s+t)^4}{r^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2\left((s+t)\left(e^{-2\delta s}+e^{-2\delta t}\right) + 2(t-s)e^{-\delta(s+t)}\right)\\
&\quad +(1-e^{-\delta(s+t)})^2\left((s-t)(1+e^{-2\delta(s+t)}) - 2(s+t)e^{-\delta(s+t)}\right)\\
&\quad +2\frac{s(s+t)^2}{r}(1-e^{-\delta(s+t)})^2\left(e^{-\delta s}-e^{-\delta t}\right)^2.
\end{aligned}
$$

Furthermore, since $s=\sqrt{x}$ and $t=\sqrt{x+r}$, we have $r=t^2-s^2=(t+s)(t-s)$. It follows

$$
\frac{(s+t)^2}{r} = \frac{(s+t)}{(t-s)} \quad \text{and} \quad \frac{(s+t)^4}{r^2} = \frac{(s+t)^2}{(t-s)^2}.
$$

Then, we have

$$
\begin{aligned}
g_{\delta,r}(x) &= \frac{(s+t)^2}{(t-s)^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2\left((s+t)\left(e^{-2\delta s}+e^{-2\delta t}\right) + 2(t-s)e^{-\delta(s+t)}\right)\\
&\quad -(1-e^{-\delta(s+t)})^2\left((t-s)(1+e^{-2\delta(s+t)}) + 2(s+t)e^{-\delta(s+t)}\right)\\
&\quad +2\frac{s(s+t)}{(t-s)}(1-e^{-\delta(s+t)})^2\left(e^{-\delta s}-e^{-\delta t}\right)^2.
\end{aligned}
$$

Moreover, from [14], Lemma 5.2, we have

$$
1-e^{-\delta(s+t)} - \frac{1}{r}(s+t)^2\left(e^{-\delta s}-e^{-\delta t}\right) > 0,
$$

hence

$$
\left(1-e^{-\delta(s+t)}\right)^2 > \frac{(s+t)^2}{(t-s)^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2. \tag{36}
$$

Then, from (36), we obtain

$$\begin{aligned}
g_{\delta,r}(x) \quad < \quad & \frac{(s+t)^2}{(t-s)^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2\left((s+t)\left(e^{-2\delta s}+e^{-2\delta t}\right)+2(t-s)e^{-\delta(s+t)}\right)\\
& -\frac{(s+t)^2}{(t-s)^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2\left((t-s)(1+e^{-2\delta(s+t)})+2(s+t)e^{-\delta(s+t)}\right)\\
& +2\frac{s(s+t)}{(t-s)}(1-e^{-\delta(s+t)})^2\left(e^{-\delta s}-e^{-\delta t}\right)^2\\
< \quad & (s+t)\left(e^{-\delta s}-e^{-\delta t}\right)^2\tilde{g}_{\delta,r}(x),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{g}_{\delta,r}(x) \quad = \quad & \frac{(s+t)}{(t-s)^2}\left((s+t)\left(e^{-2\delta s}+e^{-2\delta t}\right)+2(t-s)e^{-\delta(s+t)}\right)\\
& -\frac{(s+t)}{(t-s)^2}\left((t-s)(1+e^{-2\delta(s+t)})+2(s+t)e^{-\delta(s+t)}\right)\\
& +2\frac{s}{(t-s)}(1-e^{-\delta(s+t)})^2\\
= \quad & \frac{(s+t)^2}{(t-s)^2}\left(e^{-2\delta s}+e^{-2\delta t}\right)-2\frac{(s+t)^2}{(t-s)^2}e^{-\delta(s+t)}\\
& -\frac{(s+t)}{(t-s)}(1+e^{-2\delta(s+t)})+2\frac{(s+t)}{(t-s)}e^{-\delta(s+t)}+2\frac{s}{(t-s)}(1-e^{-\delta(s+t)})^2\\
= \quad & \frac{(s+t)^2}{(t-s)^2}\left(e^{-2\delta s}-2e^{-\delta(s+t)}+e^{-2\delta t}\right)-\frac{(s+t)}{(t-s)}\left(1-2e^{-\delta(s+t)}+e^{-2\delta(s+t)}\right)\\
& +2\frac{s}{(t-s)}\left(1-e^{-\delta(s+t)}\right)^2\\
= \quad & \frac{(s+t)^2}{(t-s)^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2-\frac{(s+t)}{(t-s)}\left(1-e^{-\delta(s+t)}\right)^2+2\frac{s}{(t-s)}\left(1-e^{-\delta(s+t)}\right)^2\\
= \quad & \frac{(s+t)^2}{(t-s)^2}\left(e^{-\delta s}-e^{-\delta t}\right)^2-\left(1-e^{-\delta(s+t)}\right)^2.
\end{aligned}$$

Then, from (36), we obtain that $\tilde{g}_{\delta,r}(x)<0$. Finally, we have

$$g_{\delta,r}(x)<(s+t)\left(e^{-\delta s}-e^{-\delta t}\right)^2\tilde{g}_{\delta,r}(x)<0,$$

from which we deduce that $h_{\delta,r}(x)<0$. □

5 Proof of the main result

If (P) has a unique classical solution, from Theorem 4.6, (8) is satisfied.
Conversely, if (8) holds, due to Theorem 4.6, it suffices to prove that system (23) has a unique solution such that (29) holds. The proof is divided in three parts. First, we will make explicit the determinant of system (23). Then, we will show the uniqueness of the solution, to this end, we will inverse the determinant with the help of functional calculus. Finally, we will prove that ψ_1 and ψ_2 have the expected regularity.

5.1 Calculus of the determinant

Now we have to make explicit the determinant. Recall system (23)

$$\left\{\begin{array}{rcrcl} \left(P_1^+ + P_1^-\right)\psi_1 & + & \left(P_2^- - P_2^+\right)\psi_2 & = & S_1 \\ M\left(P_2^- - P_2^+\right)\psi_1 & + & \left(P_3^+ + P_3^-\right)\psi_2 & = & S_2. \end{array}\right.$$

We write the previous system as a matrix equation $\Lambda\Psi = S$, where

$$\Lambda = \begin{pmatrix} P_1^+ + P_1^- & P_2^- - P_2^+ \\ M\left(P_2^- - P_2^+\right) & P_3^+ + P_3^- \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$

To solve system (23), we will study the determinant

$$\det(\Lambda) := \left(P_1^+ + P_1^-\right)\left(P_3^+ + P_3^-\right) - M\left(P_2^- - P_2^+\right)^2,$$

of the matrix Λ. We develop it to obtain

$$\det(\Lambda) = D_1^+ + D_1^- + D_2, \tag{37}$$

where

$$D_1^+ = P_1^+P_3^+ - M\left(P_2^+\right)^2, \quad D_1^- = P_1^-P_3^- - M\left(P_2^-\right)^2 \quad \text{and} \quad D_2 = P_1^+P_3^- + P_1^-P_3^+ + 2MP_2^+P_2^-.$$

Using section 3.4, we obtain

$$D(\det(\Lambda)) = D(D_1^+ + D_1^- + D_2) = D(M^3),$$

which justify the equality in (37). In the sequel, we precise the terms D_1^+ and D_1^-.

Lemma 5.1. *We have*

1. $D_1^+ = 4k_+^2(L_+ + M)^2 U_+^{-2} V_+^{-2} D^+$, *with*

$$\begin{aligned} D^+ &= L_+\left(\left(I - e^{2d(L_+ + M)}\right)^2 - \frac{1}{r_+^2}(L_+ + M)^4\left(e^{2dM} - e^{2dL_+}\right)^2\right) \\ &\quad - M\left(\left(I - e^{d(L_+ + M)}\right)^2 + \frac{1}{r_+^2}(L_+ + M)^2\left(e^{dM} - e^{dL_+}\right)^2\right)^2. \end{aligned}$$

2. $D_1^- = 4k_-^2(L_- + M)^2 U_-^{-2} V_-^{-2} D^-$, *with*

$$\begin{aligned} D^- &= L_-\left(\left(I - e^{2c(L_- + M)}\right)^2 - \frac{1}{r_-^2}(L_- + M)^4\left(e^{2cM} - e^{2cL_-}\right)^2\right) \\ &\quad - M\left(\left(I - e^{c(L_- + M)}\right)^2 + \frac{1}{r_-^2}(L_- + M)^2\left(e^{cM} - e^{cL_-}\right)^2\right)^2. \end{aligned}$$

Proof. 1. We have

$$P_1^+P_3^+ = 4k_+^2(L_+ + M)^2 L_+ U_+^{-2} V_+^{-2} D'_+,$$

where

$$\begin{aligned}
4D'_+ &= \left(U_+^2+V_+^2\right)\left(I-e^{2dM}\right)\left(I-e^{2dL_+}\right) \\
&\quad +2U_+V_+\left[\left(I-e^{d(L_++M)}\right)^2+\left(e^{dM}-e^{dL_+}\right)^2\right] \\
&= \left(U_+^2+V_+^2\right)\left[\left(I+e^{d(L_++M)}\right)^2-\left(e^{dM}+e^{dL_+}\right)^2\right] \\
&\quad +2U_+V_+\left[\left(I+e^{d(L_++M)}\right)^2+\left(e^{dM}+e^{dL_+}\right)^2\right] \\
&= (U_++V_+)^2\left(I+e^{d(L_++M)}\right)^2-(U_+-V_+)\left(e^{dM}+e^{dL_+}\right)^2.
\end{aligned}$$

Moreover

$$U_++V_+=2\left(I-e^{d(L_++M)}\right) \quad \text{and} \quad U_+-V_+=-\frac{2}{r_+}(L_++M)^2\left(e^{dM}-e^{dL_+}\right). \tag{38}$$

Then

$$D'_+=\left(I-e^{d(L_++M)}\right)^2\left(I+e^{d(L_++M)}\right)^2-\frac{1}{r_+^2}(L_++M)^4\left(e^{dM}-e^{dL_+}\right)^2\left(e^{dM}+e^{dL_+}\right)^2.$$

Furthermore, we have

$$\left(P_2^+\right)^2=4k_+^2(L_++M)^2MU_+^{-2}V_+^{-2}D''_+,$$

where

$$\begin{aligned}
4D''_+ &= \left[V_+\left(I+e^{dM}\right)\left(I-e^{dL_+}\right)+U_+\left(I-e^{dM}\right)\left(I+e^{dL_+}\right)\right]^2 \\
&= V_+^2\left(I+e^{dM}\right)^2\left(I-e^{dL_+}\right)^2+U_+^2\left(I-e^{dM}\right)^2\left(I+e^{dL_+}\right)^2 \\
&\quad +2U_+V_+\left(I-e^{2dM}\right)\left(I-e^{2dL_+}\right) \\
&= V_+^2\left[\left(I-e^{d(L_++M)}\right)+\left(e^{dM}-e^{dL_+}\right)\right]^2 \\
&\quad +U_+^2\left[\left(I-e^{d(L_++M)}\right)-\left(e^{dM}-e^{dL_+}\right)\right]^2 \\
&\quad +2U_+V_+\left[\left(I+e^{d(L_++M)}\right)^2-\left(e^{dM}+e^{dL_+}\right)^2\right] \\
&= V_+^2\left[\left(I-e^{d(L_++M)}\right)^2+2\left(I-e^{d(L_++M)}\right)\left(e^{dM}-e^{dL_+}\right)+\left(e^{dM}-e^{dL_+}\right)^2\right] \\
&\quad +U_+^2\left[\left(I-e^{d(L_++M)}\right)^2-2\left(I-e^{d(L_++M)}\right)\left(e^{dM}-e^{dL_+}\right)+\left(e^{dM}-e^{dL_+}\right)^2\right] \\
&\quad +2U_+V_+\left[\left(I+e^{d(L_++M)}\right)^2-\left(e^{dM}+e^{dL_+}\right)^2\right].
\end{aligned}$$

Hence

$$\begin{aligned}
4D''_+ &= \left(U_+^2+V_+^2\right)\left[\left(I-e^{d(L_++M)}\right)^2+\left(e^{dM}-e^{dL_+}\right)^2\right]\\
&\quad -2\left(U_+^2-V_+^2\right)\left(I-e^{d(L_++M)}\right)\left(e^{dM}-e^{dL_+}\right)\\
&\quad +2U_+V_+\left[\left(I+e^{d(L_++M)}\right)^2-\left(e^{dM}+e^{dL_+}\right)^2\right]\\
&= \left(U_+^2+V_+^2\right)\left[\left(I-e^{d(L_++M)}\right)^2+\left(e^{dM}-e^{dL_+}\right)^2\right]\\
&\quad -2\left(U_+^2-V_+^2\right)\left(I-e^{d(L_++M)}\right)\left(e^{dM}-e^{dL_+}\right)\\
&\quad +2U_+V_+\left[\left(I-e^{d(L_++M)}\right)^2-\left(e^{dM}-e^{dL_+}\right)^2\right]\\
&= (U_++V_+)^2\left(I-e^{d(L_++M)}\right)^2+(U_+-V_+)^2\left(e^{dM}-e^{dL_+}\right)^2\\
&\quad -2\left(U_+^2-V_+^2\right)\left(I-e^{d(L_++M)}\right)\left(e^{dM}-e^{dL_+}\right).
\end{aligned}$$

Using again (38), we obtain

$$\begin{aligned}
D''_+ &= \left(I-e^{d(L_++M)}\right)^4+\frac{1}{r_+^2}(L_++M)^4\left(e^{dM}-e^{dL_+}\right)^4\\
&\quad +\frac{2}{r_+}(L_++M)^2\left(I-e^{d(L_++M)}\right)^2\left(e^{dM}-e^{dL_+}\right)^2\\
&= \left(\left(I-e^{d(L_++M)}\right)^2+\frac{1}{r_+}(L_++M)^2\left(e^{dM}-e^{dL_+}\right)^2\right)^2.
\end{aligned}$$

Finally, since we have

$$D_1^+ = P_1^+P_3^+ - M\left(P_2^+\right)^2,$$

we obtain

$$D_1^+ = 4k_+^2(L_++M)^2U_+^{-2}V_+^{-2}D^+,$$

where $D^+ = L_+D'_+ - MD''_+$.

2. The result is similarly obtained by replacing respectively d, k_+ and r_+ by c, k_- and r_- in the proof above.

□

5.2 Inversion of the determinant

In this section, we prove that the determinant of system (23) is invertible with bounded inverse by using functional calculus. From the writing of D_1^+, D_1^-, given in Lemma 5.1 and the definition of D_2, we obtain:

$$D_1^+ = g_1^+(-A), \quad D_1^- = g_1^-(-A) \quad \text{and} \quad D_2 = g_2(-A),$$

where we have set, for $z \in \mathbb{C}\setminus\mathbb{R}_-$

$$\begin{cases}
g_1^+(z) &= 4k_+^2(\sqrt{z+r_+}+\sqrt{z})^2u_{d,r_+}^{-2}(z)v_{d,r_+}^{-2}(z)g_{d,r_+}(z)\\
g_1^-(z) &= 4k_-^2(\sqrt{z+r_-}+\sqrt{z})^2u_{c,r_-}^{-2}(z)v_{c,r_-}^{-2}(z)g_{c,r_-}(z)\\
g_2(z) &= k_+f_1^+(z)k_-f_3^-(z)+k_-f_1^-(z)k_+f_3^+(z)-2\sqrt{z}\,k_+f_2^+(z)k_-f_2^-(z),
\end{cases}$$

($u_{\delta,r}$, $v_{\delta,r}$, $g_{\delta,r}$, f_i^+ and f_i^- have been defined in section 4.4). So

$$\det(\Lambda) = D_1^+ + D_1^- + D_2 = f(-A), \tag{39}$$

with $f = g_1^+ + g_1^- + g_2$. Note that $f \in H(S_\theta)$, for some $\theta \in (0,\pi)$, moreover from Remark 4.8 and Lemma 4.9, for $x > 0$, we have

$$f(x) = g_1^+(x) + g_1^-(x) + g_2(x) < 0. \tag{40}$$

Let C_1, C_2 be linear operators in X. We will write $C_1 \sim C_2$ to mean that $C_1 = C_2 + \Sigma$, where Σ is a finite sum of terms of type $kL_+^l L_-^m M^n e^{\alpha L_+} e^{\beta L_-} e^{\delta M}$, where $k \in \mathbb{R}$; $l, m, n \in \mathbb{N}$; $\alpha, \beta, \delta \in \mathbb{R}_+$ with $\alpha + \beta + \delta \neq 0$. Note that Σ is a regular term in the sense:

$$\Sigma \in \mathcal{L}(X) \quad \text{with} \quad \Sigma(X) \subset D(M^\infty) := \bigcap_{k \geqslant 0} D(M^k).$$

Since $U_\pm \sim I, V_\pm \sim I$, then setting $W = U_- U_+ V_- V_+ \sim I$, we get that

$$\begin{cases} WP_1^+ \sim 2k_+(L_+ + M)L_+, & WP_1^- \sim 2k_-(L_- + M)L_- \\ WP_2^+ \sim 2k_+(L_+ + M), & WP_2^- \sim 2k_-(L_- + M) \\ WP_3^+ \sim 2k_+(L_+ + M), & WP_3^- \sim 2k_-(L_- + M). \end{cases}$$

Then

$$\begin{aligned} W^2 \det(\Lambda) &= \left(WP_1^+ WP_3^+ - M\left(WP_2^+\right)^2\right) + \left(WP_1^- WP_3^- - M\left(WP_2^-\right)^2\right) \\ &\quad + \left(WP_1^+ WP_3^- + WP_1^- WP_3^+ + 2MWP_2^- WP_2^+\right) \\ &\sim 4k_+^2(L_+ + M)^2(L_+ - M) + 4k_-^2(L_- + M)^2(L_- - M) \\ &\quad + 4k_+k_-(L_+ + M)(L_- + M)(L_+ + L_- + 2M). \end{aligned}$$

Now, due to (7), we have

$$\begin{aligned} W^2 \det(\Lambda) &\sim 4k_+^2 r_+(L_+ + M) + 4k_-^2 r_-(L_- + M) \\ &\quad + 4k_+k_-(L_+ + M)(L_- + M)(L_+ + L_- + 2M). \end{aligned}$$

We set

$$B = 4(L_+ + M)\left(k_+^2 r_+ + k_-^2 r_-(L_- + M)(L_+ + M)^{-1} + k_+k_-(L_- + M)(L_+ + L_- + 2M)\right).$$

Then, we obtain

$$\det(\Lambda) = W^{-2}\left(B + \sum_{j \in J} k_j L_+^{l_j} L_-^{m_j} M^{n_j} e^{\alpha_j L_+} e^{\beta_j L_-} e^{\delta_j M}\right), \tag{41}$$

where J is a finite set and for any $j \in J$:

$$k_j \in \mathbb{R};\ l_j, m_j, n_j \in \mathbb{N},\ \alpha_j, \beta_j, \delta_j \in \mathbb{R}_+ \quad \text{with} \quad \alpha_j + \beta_j + \delta_j \neq 0.$$

Lemma 5.2. *Operator B which is defined above is invertible with bounded inverse.*

Proof. From (H_3) we have $-L_+, -L_-, -M \in \text{BIP}\,(X, \theta/2)$ then, from [19], Theorem 5, p.443, there exists $\theta' \in [\theta/2, \pi/2)$, such that

$$-(L_- + M), -(L_+ + M), -(L_+ + M + L_- + M) \in \text{BIP}\,(X, \theta').$$

It follows from [19], property (2.7), p. 433, that $-(L_+ + M)^{-1} \in \text{BIP}\,(X, \theta')$ and since $0 \in \rho(L_- + M)$, from [19], corollary 3, p. 444, we deduce

$$\frac{k_-^2 r_-}{k_+^2 r_+}(L_- + M)(L_+ + M)^{-1} \in \text{BIP}\,(X, 2\theta').$$

Moreover, from [19], Theorem 3, p. 437, we have

$$B_1 \ := \ k_+^2 r_+ + k_-^2 r_-(L_- + M)(L_+ + M)^{-1} \in \mathrm{BIP}\,(X, 2\theta'),$$

and $0 \in \rho(B_1)$. Finally, from [19], corollary 3, p. 444, we obtain

$$B_2 \ := \ k_+ k_-(L_- + M)(L_+ + L_- + 2M) \in \mathrm{BIP}\,(X, 2\theta').$$

Then, $B_1 + B_2 \in \mathrm{BIP}\,(X, \theta'')$, for some $\theta'' \in [2\theta', \pi)$. Moreover, since $0 \in \rho(B_1)$, we deduce from [19], remark at the end of p. 445, that $0 \in \rho(B_1 + B_2)$. Since $4(L_+ + M)$ is invertible, we deduce that $B = 4(L_+ + M)(B_1 + B_2)$ is invertible with bounded inverse. $\square$

From (41) and Lemma 5.2, we deduce that

$$\det(\Lambda) = W^{-2} BF, \tag{42}$$

with

$$F = I + \sum_{j\in J} k_j B^{-1} L_+^{l_j} L_-^{m_j} M^{n_j} e^{\alpha_j L_+} e^{\beta_j L_-} e^{\delta_j M}. \tag{43}$$

For $z \in \mathbb{C} \setminus \mathbb{R}_-$, we set

$$\tilde{b}(z) = k_+^2 r_+ + k_-^2 r_- \frac{\sqrt{z + r_-} + \sqrt{z}}{\sqrt{z + r_+} + \sqrt{z}} + k_+ k_-(\sqrt{z + r_-} + \sqrt{z})(\sqrt{z + r_+} + \sqrt{z + r_-} + 2\sqrt{z}),$$

(note that, for $x > 0$, $\tilde{b}(x) > 0$) and

$$\tilde{f}(z) = 1 + \sum_{j\in J} k_j \tilde{b}(z)^{-1} \left(-\sqrt{z + r_+}\right)^{l_j} \left(-\sqrt{z + r_-}\right)^{m_j} \left(-\sqrt{z}\right)^{n_j} e^{-\alpha_j\sqrt{z+r_+}} e^{-\beta_j\sqrt{z+r_-}} e^{-\delta_j\sqrt{z}}.$$

Then, $B = 4(L_+ + M)\tilde{b}(-A)$ and $F = \tilde{f}(-A)$ and from (39) and (42), we have

$$f(-A) = \det(\Lambda) = W^{-2} B \tilde{f}(-A).$$

By construction, the link between f and $\tilde{f}$ is

$$f(z) = -4u_{d,r_+}^{-2}(z) v_{d,r_+}^{-2}(z) u_{c,r_-}^{-2}(z) v_{c,r_-}^{-2}(z)(\sqrt{z + r_+} + \sqrt{z})\tilde{b}(z)\tilde{f}(z). \tag{44}$$

Proposition 5.3. *Operator $F \in \mathcal{L}(X)$ defined above is invertible with bounded inverse.*

Proof. Note that $f, \tilde{f} \in H(S_\theta)$, for a given $\theta \in (0, \pi)$. Moreover, since for $z \in \mathbb{C} \setminus \mathbb{R}_-$, we have

$$b_j(z) = k_j \tilde{b}^{-1}(z) \left(-\sqrt{z + r_+}\right)^{l_j} \left(-\sqrt{z + r_-}\right)^{m_j} \left(-\sqrt{z}\right)^{n_j}, \quad \text{for all } j \in J,$$

are polynomial functions, we obtain $1 - \tilde{f} \in \mathcal{E}_\infty(S_\theta)$.

From (40), we know that f do not vanish on $\mathbb{R}_+ \setminus \{0\}$ and since $u_{d,r_+}, u_{c,r_-}, v_{d,r_+}, v_{c,r_-}, \tilde{b} > 0$ on $\mathbb{R}_+ \setminus \{0\}$, we deduce, from (44), that $\tilde{f}$ do not vanish on $\mathbb{R}_+ \setminus \{0\}$. Then, applying Lemma 4.7 with $G = \tilde{f}$ and $P = -A$, we deduce that $F = \tilde{f}(-A)$ is invertible with bounded inverse. $\square$

Finally, we obtain the following result

Proposition 5.4. *Operator* $\det(\Lambda)$ *is invertible with bounded inverse.*

Proof. From (42), Lemma 5.2 and Proposition 5.3, we have $\det(\Lambda) = W^{-2}BF$, which is invertible with bounded inverse. $\square$

5.3 Regularity

From Theorem 4.6, it remains to show that system (23) has a unique solution (ψ_1, ψ_2) satisfying (29). The uniqueness of the solution (ψ_1, ψ_2) is furnished by Proposition 5.4 and we get

$$\begin{cases} \psi_1 &= \left(P_3^+ + P_3^-\right)[\det(\Lambda)]^{-1} S_1 - \left(P_2^- - P_2^+\right)[\det(\Lambda)]^{-1} S_2 \\ \psi_2 &= -M\left(P_2^- - P_2^+\right)[\det(\Lambda)]^{-1} S_1 + \left(P_1^+ + P_1^-\right)[\det(\Lambda)]^{-1} S_2. \end{cases} \tag{45}$$

To obtain (29), we have first to study $[\det(\Lambda)]^{-1}$.

Lemma 5.5. *There exists $R \in \mathcal{L}(X)$ such that*

$$R(X) \subset D(M), \quad [\det(\Lambda)]^{-1} = N^{-1} + N^{-1}R,$$

where $N = 4k_+k_-(L_- + M)(L_+ + M)(L_+ + L_- + 2M)$.

Proof. From (42), we have

$$\begin{aligned} \det(\Lambda) &= W^{-2}BF \\ &= NU_-^{-2}U_+^{-2}V_-^{-2}V_+^{-2}BN^{-1}F. \end{aligned}$$

Using (12), (43) and Lemma 5.1 in [14], we get that

$$F \sim I, \quad U_+^{-1} \sim I, \quad U_-^{-1} \sim I, \quad V_+^{-1} \sim I \quad \text{and} \quad V_-^{-1} \sim I.$$

So, we have

$$\det(\Lambda) = NBN^{-1}(I + \Sigma_1),$$

where $\Sigma_1 \in \mathcal{L}(X)$ and $\Sigma_1(X) \subset D(M^\infty)$. But

$$\begin{aligned} BN^{-1} &= \left(4k_+^2 r_+\left(I + \frac{k_-^2 r_-}{k_+^2 r_+}(L_- + M)(L_+ + M)\right)M^{-3}\right)N^{-1} \\ &\quad + 4k_+k_-(L_- + M)(L_+ + M)(L_+ + L_- + 2M)N^{-1} \\ &= I + \Sigma_2, \end{aligned}$$

with $\Sigma_2 \in \mathcal{L}(X)$ and $\Sigma_2(X) \subset D(M)$. Finally

$$\det(\Lambda) = N(I + \Sigma_3),$$

where $\Sigma_3 \in \mathcal{L}(X)$ and $\Sigma_3(X) \subset D(M)$. Thus, from Lemma 5.1 in [14], we have

$$\begin{aligned} [\det(\Lambda)]^{-1} &= N^{-1}(I + \Sigma_3)^{-1} \\ &= N^{-1}(I + R), \end{aligned}$$

with $R \in \mathcal{L}(X)$ and $R(X) \subset \Sigma_3(X) \subset D(M)$. □

From (26) and (28), we deduce that

$$\begin{cases} S_1 &= -k_+(L_+ + M)\left(\tilde{\varphi_1}^+ + \tilde{\varphi_3}^+\right) - k_-(L_- + M)\left(\tilde{\varphi_1}^- + \tilde{\varphi_3}^-\right) + \tilde{R}_1 \\ S_2 &= k_+(L_+ + M)\left(\tilde{\varphi_2}^+ + \tilde{\varphi_4}^+\right) - k_-(L_- + M)\left(\tilde{\varphi_2}^- + \tilde{\varphi_4}^-\right) + \tilde{R}_2, \end{cases}$$

where $\tilde{R}_1 \in D(M)$ and $\tilde{R}_2 \in D(M^\infty)$. Using (21), (16), (8), (9), Remark 4.5 and Remark 4.3, we obtain that

$$\tilde{\varphi}_1^+, \tilde{\varphi}_1^-, \tilde{\varphi}_2^+, \tilde{\varphi}_2^-, \tilde{\varphi}_3^+, \tilde{\varphi}_3^-, \tilde{\varphi}_4^+, \tilde{\varphi}_4^- \in (D(M), X)_{2+\frac{1}{p},p}. \tag{46}$$

It follows that $S_1, S_2 \in (D(M),X)_{1+\frac{1}{p},p}$ and thus

$$\left\{\begin{array}{lllr} [\det(\Lambda)]^{-1} S_1 & = & N^{-1}(I+R)\,S_1 & \in (D(M),X)_{4+\frac{1}{p},p} \\ [\det(\Lambda)]^{-1} S_2 & = & N^{-1}(I+R)\,S_2 & \in (D(M),X)_{4+\frac{1}{p},p}. \end{array}\right. \tag{47}$$

Moreover, from (45), we have

$$\left\{\begin{array}{lll} \psi_1 & = & 2\left(k_+(L_+ + M) + k_-(L_- + M)\right)[\det(\Lambda)]^{-1} S_1 \\ & & +2\left(k_+(L_+ + M) - k_-(L_- + M)\right)[\det(\Lambda)]^{-1} S_2 + \tilde{S}_1 \\ \psi_2 & = & 2\left(k_+(L_+ + M) - k_-(L_- + M)\right)[\det(\Lambda)]^{-1} S_1 \\ & & +2\left(k_+(L_+ + M)L_+ + k_-(L_- + M)L_-\right)[\det(\Lambda)]^{-1} S_2 + \tilde{S}_2, \end{array}\right. \tag{48}$$

where $\tilde{S}_1$, $\tilde{S}_2 \in D(M^\infty)$. Finally, (47), (48) and (9) gives

$$\psi_1 \in (D(M),X)_{3+\frac{1}{p},p} = (D(A),X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \psi_2 \in (D(M),X)_{2+\frac{1}{p},p} = (D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p}.$$

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