

On the regularity of a generalized diffusion problem arising in population dynamics set in a cylindrical domain

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Abstract

In this paper, we consider a generalized diffusion problem arising in population dynamics. To this end, we study a fourth order operational equation of elliptic type, with various boundary conditions. We show existence, uniqueness and regularity of a classical solution on a cylindrical domain under some necessary and sufficient conditions on the data. This elliptic problem is solved in $L^p(a, b; X)$, $p \in (1, +\infty)$, where $(a, b) \subset \mathbb{R}$ and X is a UMD Banach space. Our techniques use essentially the functional calculus and the semigroup theory.

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1 Introduction

Many problems in biology may be described by partial differential equations. It means that the model is constructed by averaging the density of a population and keeping only time and space variables. This leads to study the population density, so-called population dynamics, governed by reaction-diffusion equations. The model considered in general is the following:

$$\partial_t u = \alpha \Delta u + f(u), \quad \text{in } \mathbb{R}_+ \times \Omega, \quad (1.1)$$

where $u(t, x)$ is the density of population at time t and position x , Ω is an open set of \mathbb{R}^d , $d \geq 2$, and f is a non-linear growth interaction. The diffusion operator term Δu is obtained from Fick's law. In 1981, D.S. Cohen and J.D. Murray [6] derived a more complete model with a biharmonic term added to the harmonic one. This model has been obtained by studying the motility of cells, for which they noticed that the classical diffusion model was not sufficient. This model is:

$$\partial_t u = k_1 \Delta u - k_2 \Delta^2 u + f(u) \quad \text{in } \mathbb{R}_+ \times \Omega, \quad (1.2)$$

where $k_1, k_2 \in \mathbb{R} \setminus \{0\}$. The biharmonic term $\Delta^2 u$ represents the long range diffusion, whereas the harmonic term Δu represents the short range diffusion. Another derivation of this model has been obtained in 1984 by F.L. Ochoa [20] who studied the dynamics of (1.2) when f is a cubic function. His work brings out the importance of the biharmonic term in the diffusion of the population.

The classical study of nonlinear problems like (1.1) or (1.2) needs first to consider the associated linearized steady problem. This first step is essential to deduce maximal regularity results for the linearized unsteady problem. From these maximal regularity results, we can obtain the existence and uniqueness of the nonlinear problem by a fixed point theorem. In the case where Ω is an open set with C^4 -boundary, the result is well-known. For problem (1.1), we can find a detailed study in [22], see section 4, p. 13 for instance. For problem (1.2), results of existence and regularity are obtained in Hilbertian case in [19] and some other results were recently obtained in [5].

In this paper, we study problem (1.2) in the linear steady case on a bounded cylindrical open set $\Omega := (a, b) \times \omega$ of \mathbb{R}^d , with ω a bounded open set of \mathbb{R}^{d-1} with C^2 -boundary and $f \in L^p(\Omega)$, $p \in (1, +\infty)$, *i.e.*

$$k_2 \Delta^2 u - k_1 \Delta u = f \quad \text{on } \Omega. \quad (1.3)$$

We follow the work of [10] for second order problems that we adapt to fourth order problems. For this, we set $k := k_1/k_2$ and we consider the Laplace operator A_0 on the Banach space $L^q(\omega)$, $q \in (1, +\infty)$, with a homogeneous Dirichlet boundary condition on ω . Thus, equation (1.3) can be formulated as the fourth order operational differential equation

$$u^{(4)}(x) + (2A_0 - kI)u''(x) + (A_0^2 - kA_0)u(x) = f(x), \quad x \in (a, b),$$

where $f \in L^p(a, b; L^q(\omega))$, $p \in (1, +\infty)$, with $u(x) := u(x, \cdot)$ and $f(x) := f(x, \cdot)$. Then, we will consider a generalization of this problem, that is

$$u^{(4)}(x) + (2A - kI)u''(x) + (A^2 - kA)u(x) = f(x), \quad x \in (a, b), \quad (1.4)$$

under various boundary conditions, with $(A, D(A))$ a BIP operator of angle $\theta \in (0, \pi)$ on a UMD space X (see section 2 below for the definitions of BIP operator and UMD spaces) and $f \in L^p(a, b; X)$. More precisely, we study (1.4) under one of the following boundary conditions:

$$\begin{cases} u(a) = \varphi_1, & u(b) = \varphi_2, \\ u''(a) = \varphi_3, & u''(b) = \varphi_4, \end{cases} \quad (\text{BC1})$$

$$\begin{cases} u'(a) = \varphi_1, & u'(b) = \varphi_2, \\ u''(a) + Au(a) = \varphi_3, & u''(b) + Au(b) = \varphi_4, \end{cases} \quad (\text{BC2})$$

$$\begin{cases} u(a) = \varphi_1, & u(b) = \varphi_2, \\ u'(a) = \varphi_3, & u'(b) = \varphi_4, \end{cases} \quad (\text{BC3})$$

or

$$\begin{cases} u'(a) = \varphi_1, & u'(b) = \varphi_2, \\ u''(a) = \varphi_3, & u''(b) = \varphi_4, \end{cases} \quad (\text{BC4})$$

where $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in X$. We obtain (see Theorem 2.2 and Theorem 2.5) the existence and uniqueness of a classical solution u of (1.4)-(BC*i*), $i = 1, 2, 3, 4$, if and only if $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are in some real interpolation spaces.

Let B_0, \dots, B_{n-1} be linear operators in X , we recall that u is a classical solution (in $L^p(a, b; X)$) of a differential equation

$$u^{(n)}(x) + \sum_{j=0}^{n-1} B_j u^{(j)}(x) = f(x), \quad x \in (a, b), \quad (1.5)$$

if u is a solution of (1.5) with the following regularity

$$u \in W^{n,p}(a, b; X) \quad \text{and} \quad x \mapsto B_j u^{(j)}(x) \in L^p(a, b; X), \quad j = 0, \dots, n-1.$$

Similarly, u is a classical solution of problem (1.5)-(BC), where (BC) are some boundary conditions if u is a classical solution of (1.5) and satisfies (BC).

The paper is organised as follows. In section 2, first we detail our assumptions on space X and operator A and we describe our results. First, we state the existence and the uniqueness of the solution of (1.4)-(BC1) in Theorem 2.2. This allows us to establish in Proposition 2.3, a general representation formula for classical solutions of (1.4). Then, we state a result about the regularity of a difference of two analytic semigroups (Theorem 2.4) which is an interesting result in itself and constitutes an important tool to prove the main result (Theorem 2.5). We then present Theorem 2.5, completing Theorem 2.2, which gives existence and uniqueness of a classical solution u of (1.4) under one of the boundary conditions (BC2), (BC3) or (BC4). As a consequence of this theorem, we obtain the Corollary 2.7 which states existence and uniqueness of a solution u in $W^{4,p}(\Omega)$ of (1.3). In section 3, we prove Theorem 2.2 and Proposition 2.3. Section 4 is devoted to the proof of Theorem 2.4. Finally, in the last section, we prove Theorem 2.5.

2 Assumptions and statement of results

In all the paper, $(X, \|\cdot\|)$ is a complex Banach space. Recall (see, for instance, [14]) that a closed linear operator $T \in X$ is called sectorial of angle $\alpha \in (0, \pi)$ if

- i) $\sigma(T) \subset \overline{S_\alpha}$,
- ii) $\forall \alpha' \in (\alpha, \pi), \quad \sup \{ \|\lambda(\lambda I - T)^{-1}\|_{\mathcal{L}(X)} : \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}} \} < \infty,$

where

$$S_\alpha := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \alpha\}. \quad (2.1)$$

It is known that any injective sectorial operator T admits imaginary powers T^{is} , $s \in \mathbb{R}$, but, in general, T^{is} is not bounded (see [15], p. 342). Let $\theta \in [0, \pi)$. We denote by $\text{BIP}(X, \theta)$ (see [23]), the class of sectorial injective operators T such that

- i) $\overline{D(T)} = \overline{R(T)} = X$,
- ii) $\forall s \in \mathbb{R}, T^{is} \in \mathcal{L}(X)$,
- iii) $\exists C \geq 0, \forall s \in \mathbb{R}, \|T^{is}\|_{\mathcal{L}(X)} \leq Ce^{|s|\theta}$.

All the results stated in this paper use the well-known Dore-Venni theorem (see [8] and its generalization [23]), which needs to consider a UMD space X . Recall that a Banach space X is a UMD space if and only if for some $p \in (1, +\infty)$ (and thus for all p) the Hilbert transform is bounded from $L^p(\mathbb{R}, X)$ into itself (see [1, 3, 4]).

In all the sequel, $k \in \mathbb{R} \setminus \{0\}$, A denotes a linear operator in X and we assume:

- (H₁) X is a UMD space,
- (H₂) $0 \in \rho(A)$,
- (H₃) $-A \in \text{BIP}(X, \theta)$ for some $0 < \theta < \pi$,
- (H₄) $[k, +\infty) \subset \rho(A)$.

For some of our results, we need a supplementary assumption:

$$(H_5) \quad \sigma(A) \subset (-\infty, 0) \quad \text{and} \quad \forall \theta \in (0, \pi), \sup_{\lambda \in S_\theta} \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} < +\infty,$$

which means that $-A$ is a sectorial operator of any angle $\theta \in (0, \pi)$. Let us give some remarks on our assumptions.

Remark 2.1.

1. Assumption (H₄) is relevant only if $k < 0$, since for $k > 0$, (H₂) and (H₃) imply (H₄).
2. The case $k = 0$ has been stated (with particular boundary conditions) in [9]. The study is different from ours and requires a particular representation formula for the solution.
3. To solve (1.4) in the scalar case (with $-A > 0$), it is necessary to introduce the roots $\pm\sqrt{-A+k}$, $\pm\sqrt{-A}$ of the characteristic equation

$$r^4 + (2A - k)r^2 + (A^2 - kA)r = 0,$$

this is why, in our operational case, we consider the operators

$$L := -\sqrt{-A+k}I \quad \text{and} \quad M := -\sqrt{-A}. \quad (2.2)$$

Due to (H₃) and (H₄), $-A$ and $-A+kI$ are sectorial operators, so the existence of L and M is ensured (see for instance [14]).

4. Applying Proposition 3.1.9 in [14], we have $D(L) = D(M)$. Thus, for $k, l \in \mathbb{N}$ and $l \leq k$

$$D(L^k) = D(M^k) = D(L^l M^{k-l}) = D(M^l L^{k-l}).$$

5. Due to (H₃) and (H₄), $-A+kI \in \text{BIP}(X, \theta)$ (see [2], Theorem 2.3, p. 69), from which we deduce

$$-L, -M \in \text{BIP}(X, \theta/2),$$

(see [14], Proposition 3.2.1, e), p. 71). Since $0 < \theta/2 < \pi/2$, L and M generate bounded analytic semigroups $(e^{xL})_{x \geq 0}$ and $(e^{xM})_{x \geq 0}$ (see [23], Theorem 2, p. 437).

6. From (H₂) and (H₄), we deduce that $0 \in \rho(M) \cap \rho(L)$ (see [14], Proposition 3.3.1, e), p. 62). Thus, assumptions (H₁), (H₂) and (H₃) lead us to apply the Dore-Venni theorem (see [8]) to obtain $0 \in \rho(L+M)$.

7. Definitions in (2.2) imply that

$$\forall \psi \in D(L^2) = D(M^2), \quad (L^2 - M^2)\psi = k\psi. \quad (2.3)$$

Moreover $L - M = (L - M)(L + M)(L + M)^{-1}$, so

$$\forall \psi \in D(L), \quad (L - M)\psi = k(L + M)^{-1}\psi. \quad (2.4)$$

To determine the conditions on $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in X$ characterizing the existence of a unique classical solution of (1.4)-(BC1), (1.4)-(BC2), (1.4)-(BC3) and (1.4)-(BC4), we need to introduce interpolation spaces. To this end, we denote $(Y, X)_{\theta, q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$, the real interpolation spaces (see, for instance, [13, 16]) between a subspace $Y \subset X$ and X . Moreover, we need to introduce the space $(D(A), X)_{\theta, q}$ with $\theta > 1$; to this end, we will use the following notation:

$$(D(A), X)_{k+\theta', q} := \{\psi \in D(A^k) : A^k\psi \in (D(A), X)_{\theta', q}\}, \quad k \in \mathbb{N}, \theta' \in (0, 1).$$

First, we solve problem (1.4)-(BC1) whose boundary conditions are well adapted to the equation and which can be easily treated by considering two second order problems. We obtain:

Theorem 2.2. *Let $f \in L^p(a, b; X)$ with $a, b \in \mathbb{R}$, $a < b$ and $p \in (1, \infty)$. Assume that (H_1) , (H_2) , (H_3) and (H_4) hold. Then, there exists a unique classical solution of (1.4) – (BC1) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}.$$

This unique classical solution is denoted by $F_{\Phi, f}$ with $\Phi := (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ and is explicitly described by

$$\begin{aligned} F_{\Phi, f}(x) &= e^{(x-a)M} Z\varphi_1 + e^{(b-x)M} Z\varphi_2 \\ &\quad - \frac{1}{2} e^{(x-a)M} ZM^{-1} \int_a^b e^{(s-a)M} v_0(s) ds - \frac{1}{2} e^{(b-x)M} ZM^{-1} \int_a^b e^{(b-s)M} v_0(s) ds \\ &\quad + \frac{1}{2} M^{-1} \int_a^x e^{(x-s)M} v_0(s) ds + \frac{1}{2} M^{-1} \int_x^b e^{(s-x)M} v_0(s) ds \\ &\quad - e^{(b-x)M} e^{(b-a)M} \varphi_1 - e^{(x-a)M} e^{(b-a)M} \varphi_2 \\ &\quad + \frac{1}{2} e^{(x-a)M} Z e^{(b-a)M} M^{-1} \int_a^b e^{(b-s)M} v_0(s) ds \\ &\quad + \frac{1}{2} e^{(b-x)M} Z e^{(b-a)M} M^{-1} \int_a^b e^{(s-a)M} v_0(s) ds, \quad x \in [a, b], \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} v_0(x) &:= e^{(x-a)L} W(\varphi_3 + A\varphi_1) + e^{(b-x)L} W(\varphi_4 + A\varphi_2) \\ &\quad - \frac{1}{2} e^{(x-a)L} WL^{-1} \int_a^b e^{(s-a)L} f(s) ds - \frac{1}{2} e^{(b-x)L} WL^{-1} \int_a^b e^{(b-s)L} f(s) ds \\ &\quad + \frac{1}{2} L^{-1} \int_a^x e^{(x-s)L} f(s) ds + \frac{1}{2} L^{-1} \int_x^b e^{(s-x)L} f(s) ds \\ &\quad - e^{(b-x)L} e^{(b-a)L} (\varphi_3 + A\varphi_1) - e^{(x-a)L} e^{(b-a)L} (\varphi_4 + A\varphi_2) \\ &\quad + \frac{1}{2} e^{(x-a)L} W e^{(b-a)L} L^{-1} \int_a^b e^{(b-s)L} f(s) ds \\ &\quad + \frac{1}{2} e^{(b-x)L} W e^{(b-a)L} L^{-1} \int_a^b e^{(s-a)L} f(s) ds, \quad x \in [a, b], \end{aligned} \quad (2.6)$$

$$Z := (I - e^{2(b-a)M})^{-1} \quad \text{and} \quad W := (I - e^{2(b-a)L})^{-1}.$$

The existence of Z and W are proved in virtue of (5.9) in subsection 5.1.

Under other boundary conditions, equation (1.4) is more complicated to study. We need two results which are important by themselves. The first one is a representation formula for the classical solution of equation (1.4).

Proposition 2.3. *If u is a classical solution of (1.4), then there exist $K_i \in X$, $i = 1, 2, 3, 4$, such that for any $x \in [a, b]$*

$$u(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 + e^{(x-a)L}K_3 + e^{(b-x)L}K_4 + F_{0,f}(x), \quad (2.7)$$

where $F_{0,f}$ is defined in Theorem 2.2.

We state below the second result which concerns the regularity of the difference of two analytic semigroups.

Theorem 2.4. *Assume that (H_1) , (H_2) , (H_3) and (H_4) hold. For any $\psi \in X$ and $x \in [a, b]$, we set*

$$u_\psi(x) := \left(e^{(x-a)L} - e^{(x-a)M} \right) \psi \quad \text{and} \quad v_\psi(x) := \left(e^{(b-x)L} - e^{(b-x)M} \right) \psi.$$

Then, we have

1. $u_\psi \in W^{2,p}(a, b; X) \cap L^p(a, b; D(M^2))$ and $u'_\psi \in L^p(a, b; D(M))$.
2. $u_\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4))$ if and only if $\psi \in (D(M), X)_{1+\frac{1}{p}, p}$.
Furthermore, in this case, $u''_\psi \in L^p(a, b; D(M^2))$.
3. Statements 1. and 2. hold true if we replace u_ψ by v_ψ (it suffices to write $v_\psi(x) = u_\psi(b + a - x)$).

Note that if $\psi \in (D(M), X)_{3+\frac{1}{p}, p}$ then

$$e^{(\cdot-a)L}\psi, e^{(\cdot-a)M}\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4)),$$

(see Remark 4.2 below), from which we deduce that

$$u_\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4)),$$

but our result is more precise since $(D(M), X)_{3+\frac{1}{p}, p} \subset (D(M), X)_{1+\frac{1}{p}, p}$.

We are now in position to state our main result which completes Theorem 2.2:

Theorem 2.5. *Let $f \in L^p(a, b; X)$ with $a < b$, $a, b \in \mathbb{R}$ and $p \in (1, \infty)$. Assume that (H_1) , (H_2) , (H_3) and (H_4) hold. Then*

1. *there exists a unique classical solution u of (1.4) – (BC2) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}.$$

If in addition, (H_5) holds, then

2. *there exists a unique classical solution u of (1.4) – (BC3) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p},$$

3. *there exists a unique classical solution u of (1.4) – (BC4) if and only if*

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}.$$

Remark 2.6.

1. The result for the boundary conditions (BC1) can be directly generalized to the following boundary conditions:

$$\begin{cases} u(a) = \varphi_1, & u(b) = \varphi_2, \\ u''(a) + Au(a) = \varphi_3, & u''(b) + Au(b) = \varphi_4. \end{cases} \quad (\text{BC5})$$

2. In the same way, we can obtain similar results for mixed boundary conditions.

From Theorem 2.5, we deduce regularity results for equation (1.3) under some particular boundary conditions (other conditions can be considered). Let us explicit Theorem 2.2 for instance, when we consider problem (1.3)-(BC1) with $A := A_0$.

Corollary 2.7. Consider a cylindrical domain $\Omega := (a, b) \times \omega$ of \mathbb{R}^d , where $d \geq 2$, $a, b \in \mathbb{R}$, $a < b$ and ω is a bounded open set of \mathbb{R}^{d-1} with C^2 -boundary. Let $f \in L^p(\Omega)$ with $p \in (1, +\infty)$ and $p > \frac{d}{2}$; let $k_1, k_2 \in \mathbb{R} \setminus \{0\}$ such that $k_1 > k_2 C_\omega$, where $C_\omega > 0$ is the Poincaré constant in ω . Then, there exists a unique solution $u \in W^{4,p}(\Omega)$ of

$$\left\{ \begin{array}{ll} -k_1 \Delta u(x, y) + k_2 \Delta^2 u(x, y) = f(x, y), & (x, y) \in \Omega \\ u(x, y) = \Delta_y u(x, y) = 0, & (x, y) \in (a, b) \times \partial\omega \\ u(a, y) = \varphi_1(y), & y \in \omega \\ u(b, y) = \varphi_2(y), & y \in \omega \\ \Delta_y u(a, y) = \varphi_3(y), & y \in \omega \\ \Delta_y u(b, y) = \varphi_4(y), & y \in \omega, \end{array} \right. \quad (2.8)$$

if and only if

$$\varphi_1, \varphi_2 \in W^{2,p}(\omega) \cap W_0^{1,p}(\omega) \quad \text{and} \quad \Delta\varphi_1, \Delta\varphi_2, \varphi_3, \varphi_4 \in W^{2-\frac{1}{p},p}(\omega) \cap W_0^{1,p}(\omega).$$

Proof. Set $X := L^p(\omega)$ and

$$\left\{ \begin{array}{ll} D(A_0) & := W^{2,p}(\omega) \cap W_0^{1,p}(\omega) \\ (A_0\psi)(y) & := \Delta_y \psi(y). \end{array} \right.$$

With these notations, problem (2.8) becomes (1.4) with the boundary conditions (BC1). From [25] (Proposition 3, p. 207), X satisfies (H_1) and from [11] (Theorem 9.15 and Lemma 9.17), A_0 satisfies (H_2) . Moreover, (H_3) is satisfied from [24] (Theorem C, p. 166-167). Since C_ω is the Poincaré constant in ω , we have $(C_\omega, +\infty) \subset \rho(A_0)$. Moreover, $k := k_1/k_2 > C_\omega$, thus (H_4) is satisfied.

Finally, all the assumptions of Theorem 2.2 are satisfied. It follows that, there exists a unique classical solution of (1.4)-(BC1) if and only if

$$\varphi_1, \varphi_2 \in (D(A_0), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A_0), X)_{\frac{1}{2p},p}. \quad (2.9)$$

Now, it remains to show that if $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ satisfy (2.9), then the classical solution u satisfies $u \in W^{4,p}(\Omega)$. At this end, we will make explicit the interpolation spaces that appear in (2.9). We have

$$(D(A_0), X)_{\frac{1}{2p},p} = \left(W^{2,p}(\omega) \cap W_0^{1,p}(\omega), L^p(\omega) \right)_{\frac{1}{2p},p},$$

and from [13] (p. 683, proposizione 3 and p. 681, 1.10) and [26] (p. 317, Theorem 1), since $2 - \frac{1}{p} > 1$ is never integer, we have

$$(W^{2,p}(\omega), L^p(\omega))_{\frac{1}{2p},p} = W^{2-\frac{1}{p},p}(\omega). \quad (2.10)$$

Set $\nu := 2 - \frac{1}{p} - \frac{d-1}{p} = 2 - \frac{d}{p}$. Since $p > \frac{d}{2}$, we have $\nu > 0$. From the Sobolev embedding theorem (see [26], section 4.6.1, p. 327-328), we have:

$$W^{2-\frac{1}{p},p}(\omega) \hookrightarrow C(\bar{\omega}).$$

Thus, the traces of the elements of the space described in (2.10) are well defined. From [12] (Proposition 5.9, p. 334) and [26] (section 4.3.3, Theorem, p. 321), we deduce that

$$(D(A_0), X)_{\frac{1}{2p},p} = \left\{ \psi \in W^{2-\frac{1}{p},p}(\omega) : \psi = 0 \text{ on } \partial\omega \right\},$$

and

$$\begin{aligned} (D(A_0), X)_{1+\frac{1}{2p},p} &= \left\{ \psi \in D(A_0) : A_0\psi \in W^{2-\frac{1}{p},p}(\omega) \text{ and } \Delta\psi = 0 \text{ on } \partial\omega \right\} \\ &= \left\{ \psi \in W^{2,p}(\omega) : \Delta\psi \in W^{2-\frac{1}{p},p}(\omega) \text{ and } \psi = \Delta\psi = 0 \text{ on } \partial\omega \right\}. \end{aligned}$$

Now, the classical solution u of

$$u^{(4)}(x) + (2A_0 - kI)u''(x) + (A_0^2 - kA_0)u(x) = f(x), \quad a.e. x \in (a, b),$$

satisfies

$$\begin{cases} u \in W^{4,p}(a, b; L^p(\omega)) \\ x \mapsto A_0 u''(x) \in L^p(a, b; L^p(\omega)) \\ x \mapsto A_0^2 u(x) \in L^p(a, b; L^p(\omega)), \end{cases}$$

and

$$\begin{cases} u(a, y) = \varphi_1(y), & u(b, y) = \varphi_2(y), & y \in \omega \\ \Delta_y u(a, y) = \varphi_3(y), & \Delta_y u(b, y) = \varphi_4(y), & y \in \omega \\ u(x, \zeta) = \Delta_y u(x, \zeta) = 0, & & x \in (a, b) \text{ and } \zeta \in \partial\omega. \end{cases}$$

Since

$$\begin{cases} u \in W^{2,p}(a, b; L^p(\omega)) \cap L^p(a, b; W^{2,p}(\omega)) \cap W_0^{1,p}(\omega) \\ u(x, \zeta) = 0, \quad \zeta \in \partial\omega \\ \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0, \quad \text{on } \partial\omega, \end{cases}$$

we can apply on $u(x, \cdot)$, $x \in (a, b)$, in virtue of the Sobolev extension theorem on \mathbb{R}^{d-1} , an extension operator which maps continuously $L^p(\omega)$ into $L^p(\mathbb{R}^{d-1})$ and $W^{2,p}(\omega)$ into $W^{2,p}(\mathbb{R}^{d-1})$. Then, from the Mihlin's theorem (see [18]), we deduce that

$$u \in W^{2,p}((a, b) \times \omega) = W^{2,p}(\Omega).$$

By reiterating the same arguments to the other regularities, we obtain that $u \in W^{4,p}(\Omega)$. \square

Equation (1.3) can be studied with other boundary conditions taking into account the results of Theorem 2.5. We can also obtain anisotropic results by considering $f \in L^p(a, b; L^q(\omega))$ with $p, q \in (1, +\infty)$.

3 Proofs of preliminary results

First, we recall some classical trace results.

Remark 3.1.

1. Let T be a closed linear operator in X and $k \in \mathbb{N} \setminus \{0\}$. From [13] p. 677 and [16] p. 39, if

$$u \in W^{n,p}(a, b; X) \cap L^p(a, b; D(T^k)),$$

where $n \in \mathbb{N} \setminus \{0\}$, then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $c \in \{a, b\}$, we have

$$u^{(j)}(c) \in (D(T^k), X)_{\frac{j}{n} + \frac{1}{np}, p}. \quad (3.1)$$

Moreover, recall that, if we denote $(X, D(T^k))_{1-\theta, p} := (D(T^k), X)_{\theta, p}$, where $k \in \mathbb{N} \setminus \{0\}$ and $\theta \in (0, 1)$ is such that $k\theta \notin \mathbb{N}$, from reiteration theorem (see [13, 16]), we have

$$\begin{aligned} (D(T^k), X)_{\theta, p} &= (X, D(T^k))_{1-\theta, p} = (X, D(T))_{k-k\theta, p} \\ &= (X, D(T))_{(k-1)+1-k\theta, p} = (D(T), X)_{(k-1)+k\theta, p}, \end{aligned} \quad (3.2)$$

which implies that

$$(D(T^k), X)_{\theta, p} \subset D(T^{k-1}).$$

In particular, from (3.2), if $k\theta < 1$, the previous inclusion is always satisfied.

2. Let $u \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2))$. Then, from (3.1), for any $c \in \{a, b\}$, we obtain

$$u(c) \in (D(A^2), X)_{\frac{1}{4p}, p}, \quad u'(c) \in (D(A^2), X)_{\frac{1}{4} + \frac{1}{4p}, p} \quad \text{and} \quad u''(c) \in (D(A^2), X)_{\frac{1}{2} + \frac{1}{4p}, p}.$$

From (3.2), we deduce, for any $c \in \{a, b\}$

$$u(c) \in (D(A), X)_{1+\frac{1}{2p}, p}, \quad u'(c) \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad u''(c) \in (D(A), X)_{\frac{1}{2p}, p}. \quad (3.3)$$

3.1 Proof of Theorem 2.2

Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in X$ be such that

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}. \quad (3.4)$$

From (3.4), there exists (see [10], Theorem 5, p. 173) a unique classical solution

$$v_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A)),$$

of

$$\begin{cases} v''(x) + (A - kI)v(x) = f(x), & a.e. x \in (a, b) \\ v(a) = \varphi_3 + A\varphi_1 \\ v(b) = \varphi_4 + A\varphi_2. \end{cases} \quad (3.5)$$

Then, since $\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \subset (D(A), X)_{\frac{1}{2p}, p}$, there exists a unique classical solution

$$u_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A)),$$

of

$$\begin{cases} u''(x) + Au(x) = v_0(x), & a.e. x \in (a, b) \\ u(a) = \varphi_1 \\ u(b) = \varphi_2. \end{cases} \quad (3.6)$$

A simple computation shows that u_0 satisfies (1.4)-(BC1). It remains to show that u_0 has the expected regularity. Let $w_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A))$ be the unique classical solution of

$$\begin{cases} w''(x) + Aw(x) = Av_0(x), & a.e. x \in (a, b) \\ w(a) = A\varphi_1 \\ w(b) = A\varphi_2. \end{cases}$$

Then $A^{-1}w_0$ is a classical solution of (3.6). So, by uniqueness of the solution, we obtain

$$u_0 = A^{-1}w_0 \in L^p(a, b; D(A^2)).$$

Moreover, from (3.6), we have

$$u_0'' = v_0 - Au_0 = v_0 - w_0 \in W^{2,p}(a, b; X) \cap L^p(a, b; D(A)).$$

Hence u_0 is a classical solution of (1.4)-(BC1). Uniqueness is straightforward since if u is a classical solution of (1.4)-(BC1) then $v := u'' + Au$ is a classical solution of (3.5). By uniqueness, $v = v_0$, so u is a classical solution of (3.6) and again, by uniqueness, $u = u_0$.

Conversely, if there exists a classical solution of (1.4)-(BC1), then (3.4) holds (see (3.3)).

The representation formula (2.5) of the classical solution $F_{\Phi, f}$ of (1.4)-(BC1) is obtained in the following manner. We notice that $v_0 := F_{\Phi, f}'' + AF_{\Phi, f}$ is the classical solution of (3.5), and then v_0 is given by (2.6) (see [10], p. 169-170). Then, since $F_{\Phi, f}$ is the classical solution of (3.6), we use again [10] to obtain (2.5).

3.2 Proof of Proposition 2.3 (Representation formula)

Let u be a classical solution of (1.4). By Theorem 2.2, we can consider the classical solution $F_{0, f}$ of (1.4)-(BC1); *i.e.* which satisfies

$$F_{0, f}(a) = F_{0, f}(b) = F_{0, f}''(a) = F_{0, f}''(b) = 0. \quad (3.7)$$

We set $u_{hom} := u - F_{0, f}$. Then, u_{hom} is a classical solution of

$$u_{hom}^{(4)}(x) + (2A - kI)u_{hom}''(x) + (A^2 - kA)u_{hom}(x) = 0, \quad a.e. x \in (a, b).$$

It implies that

$$u_{hom}^{(4)}(x) = (L^2 + M^2)u_{hom}''(x) - L^2M^2u_{hom}(x), \quad a.e. x \in (a, b). \quad (3.8)$$

So it remains to show the existence of $K_i \in X$, for $i = 1, 2, 3, 4$ such that

$$u_{hom}(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 + e^{(x-a)L}K_3 + e^{(b-x)L}K_4, \quad a.e. x \in (a, b). \quad (3.9)$$

Since u and $F_{0,f}$ are classical solutions of (1.4), u_{hom} is also a classical solution, so

$$u_{hom} \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4)) \quad \text{and} \quad u''_{hom} \in L^p(a, b; D(M^2)).$$

Thus, for almost every $x \in (a, b)$, we can set

$$\begin{cases} v(x) & := k^{-1}L^2u_{hom}(x) - k^{-1}u''_{hom}(x) \in L^p(a, b; D(M^2)) \\ w(x) & := -k^{-1}M^2u_{hom}(x) + k^{-1}u''_{hom}(x) \in L^p(a, b; D(L^2)). \end{cases} \quad (3.10)$$

From (2.3), we obtain

$$v(x) + w(x) = k^{-1}(L^2 - M^2)u_{hom}(x) = u_{hom}(x), \quad a. e. x \in (a, b). \quad (3.11)$$

Moreover, from (3.8), for almost every $x \in (a, b)$, we deduce that

$$\begin{aligned} L^{-2}v''(x) &= k^{-1}u''_{hom}(x) - k^{-1}L^{-2}u_{hom}^{(4)}(x) \\ &= k^{-1}u''_{hom}(x) - k^{-1}L^{-2}((L^2 + M^2)u''_{hom}(x) - L^2M^2u_{hom}(x)) \\ &= M^2L^{-2}v(x), \end{aligned}$$

and in the same way

$$M^{-2}w''(x) = -k^{-1}u''_{hom}(x) + k^{-1}M^{-2}u_{hom}^{(4)}(x) = L^2M^{-2}w(x).$$

Thus, v and w are, respectively, solutions of

$$v''(x) - M^2v(x) = 0 \quad \text{and} \quad w''(x) - L^2w(x) = 0, \quad a. e. x \in (a, b).$$

The previous functional equalities are set in $L^p(a, b; X)$ and make sense due to (3.8). By using the results in [10], we obtain

$$v(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 \quad \text{and} \quad w(x) = e^{(x-a)L}K_3 + e^{(b-x)L}K_4, \quad x \in (a, b). \quad (3.12)$$

Then, from (3.11) and (3.12), we deduce (3.9).

4 Proof of Theorem 2.4

The proof is essentially based on analytic semigroups results (see [17]) and interpolation spaces (see, for instance, [7] p. 381-386 and [26], Theorem p. 96) and on a corollary of the well-known Dore-Venni theorem in [8]. We recall below these results as lemmas for the reader convenience.

Lemma 4.1 ([7, 26]). *Let $\psi \in X$ and T be a generator of an analytic semigroup in X . Then, for any $n \in \mathbb{N} \setminus \{0\}$, the two next properties are equivalent:*

1. $x \mapsto T^n e^{(x-a)T} \psi \in L^p(a, b; X)$,
2. $\psi \in (D(T^n), X)_{\frac{1}{np}, p}$.

Remark 4.2. Under the assumptions of Lemma 4.1, from (3.2) it follows that, for any $n \in \mathbb{N} \setminus \{0\}$, the two next properties are equivalent:

1. $x \mapsto e^{(x-a)T} \psi \in W^{n,p}(a, b; X) \cap L^p(a, b; D(T^n))$,
2. $\psi \in (D(T), X)_{n-1+\frac{1}{p}, p}$.

Lemma 4.3 (see [8]). *Let $-T \in BIP(X, \theta)$ with $\theta \in (0, \pi/2)$, and $g \in L^p(a, b; X)$. Then, for almost every $x \in (a, b)$, we have*

$$\int_a^x e^{(x-s)T} g(s) ds \in D(T) \quad \text{and} \quad \int_x^b e^{(s-x)T} g(s) ds \in D(T).$$

Moreover,

$$x \mapsto T \int_a^x e^{(x-s)T} g(s) ds \in L^p(a, b; X) \quad \text{and} \quad x \mapsto T \int_x^b e^{(s-x)T} g(s) ds \in L^p(a, b; X).$$

Remark 4.4. The assumption that X is a UMD space is not necessary in Lemma 4.1, but is essential in Lemma 4.3.

Now, we state the proof of Theorem 2.4.

1. Let $\psi \in X$. Recall that

$$\forall x > a, \quad u_\psi(x) := \left(e^{(x-a)L} - e^{(x-a)M} \right) \psi.$$

Since, for $T := L$ or M , $(e^{xT})_{x \geq 0}$ is an analytic semigroup, we have (see, for instance, [17], Proposition 2.1.1, p. 35)

$$x \mapsto e^{(x-a)T} \psi \in C^\infty((a, b]; X) \cap C^0([a, b]; X), \quad (4.1)$$

and for any $x > a$,

$$e^{(x-a)L} \psi \in D(L^\infty) = D(M^\infty) \quad \text{and} \quad e^{(x-a)M} \psi \in D(M^\infty), \quad (4.2)$$

where

$$D(M^\infty) = D(L^\infty) := \bigcap_{k \geq 0} D(M^k) = \bigcap_{k \geq 0} D(L^k). \quad (4.3)$$

From (4.1) and (4.2), we deduce

$$\forall x > a, \quad u_\psi \in C^\infty((a, b]; X) \cap C^0([a, b]; X) \quad \text{and} \quad u(x) \in D(M^\infty).$$

Then, we obtain

$$\begin{aligned} \forall x > a, \quad u'_\psi(x) &= \left(L e^{(x-a)L} - M e^{(x-a)M} \right) \psi \\ &= M \left(e^{(x-a)L} - e^{(x-a)M} \right) \psi + (L - M) e^{(x-a)L} \psi, \end{aligned}$$

and from (2.4), we deduce

$$\forall x > a, \quad u'_\psi(x) = M u_\psi(x) + k(L + M)^{-1} e^{(x-a)L} \psi.$$

Thus, $u_\psi \in C^1((a, b]; X) \cap C^0([a, b]; X)$ is a solution of the Cauchy problem

$$\begin{cases} u'(x) = M u(x) + k(L + M)^{-1} e^{(x-a)L} \psi, & \text{a.e. } x \in (a, b], \\ u(a) = 0. \end{cases}$$

Hence, (see [21], Corollary 2.2 p. 106), u is given by the variation of constant formula:

$$\begin{aligned} \forall x \in (a, b], \quad u_\psi(x) &= \int_a^x e^{(x-s)M} k(L + M)^{-1} e^{(s-a)L} \psi \, ds \\ &= k(L + M)^{-1} M^{-1} \left[M \int_a^x e^{(x-s)M} e^{(s-a)L} \psi \, ds \right]. \end{aligned} \quad (4.4)$$

Since

$$s \mapsto e^{(s-a)L} \psi \in C^0([a, b]; X) \subset L^p(a, b; X),$$

from Lemma 4.3, we have

$$g_\psi : x \mapsto M \int_a^x e^{(x-s)M} e^{(s-a)L} \psi \, ds \in L^p(a, b; X).$$

Then, since $M^2(L + M)^{-1}M^{-1} \in \mathcal{L}(X)$, we deduce that

$$x \mapsto M^2 u_\psi(x) = k M^2 (L + M)^{-1} M^{-1} g_\psi(x) \in L^p(a, b; X).$$

We also have $u_\psi \in W^{1,p}(a, b; X)$ because $u_\psi \in C^1((a, b]; X) \cap C^0([a, b]; X)$ and

$$u'_\psi(\cdot) = M u_\psi(\cdot) + k(L + M)^{-1} e^{(\cdot-a)L} \psi \in L^p(a, b; X).$$

Furthermore,

$$x \mapsto Mu'_\psi(x) = M^2u_\psi(x) + kM(L+M)^{-1}e^{(x-a)L}\psi \in L^p(a, b; X).$$

so $u'_\psi \in L^p(a, b; D(M))$. For a. e. $x \in (a, b]$, since $e^{(x-a)L}\psi \in D(M^2)$, from (2.3) we obtain

$$\begin{aligned} u''_\psi(x) &= (L^2e^{(x-a)L} - M^2e^{(x-a)L})\psi \\ &= M^2(e^{(x-a)L} - e^{(x-a)L})\psi + (L^2 - M^2)e^{(x-a)L}\psi \\ &= M^2u_\psi(x) + kM(L+M)^{-1}e^{(x-a)L}\psi. \end{aligned} \quad (4.5)$$

Thus $u''_\psi \in L^p(a, b; X)$, hence $u_\psi \in W^{2,p}(a, b; X)$.

2. *First step.* Assume that $\psi \in (D(M), X)_{1+\frac{1}{p}, p}$. We set $\varphi := M\psi \in (D(M), X)_{\frac{1}{p}, p}$. Thus, equality (4.4), for a. e. $x \in (a, b)$, yields

$$\begin{aligned} M^3u_\varphi(x) &= kM(L+M)^{-1}M^2M^{-1} \left[M \int_a^x e^{(x-s)L}e^{(s-a)L}\varphi ds \right] \\ &= kM(L+M)^{-1}ML^{-1} \left[M \int_a^x e^{(x-s)L}Le^{(s-a)L}\varphi ds \right]. \end{aligned} \quad (4.6)$$

Since $\varphi = M\psi \in (D(M), X)_{\frac{1}{p}, p}$, from Lemma 4.1, we have

$$s \mapsto Le^{(s-a)L}\varphi \in L^p(a, b; X).$$

Then, from Lemma 4.3, we deduce

$$x \mapsto M \int_a^x e^{(x-s)L}Le^{(s-a)L}\varphi ds \in L^p(a, b; X).$$

Hence, from (4.6) and the fact that $M(L+M)^{-1}ML^{-1} \in \mathcal{L}(X)$, we obtain

$$x \mapsto M^4u_\psi(x) = M^3u_\varphi \in L^p(a, b; X). \quad (4.7)$$

It follows

$$u_\psi \in L^p(a, b; D(M^4)) \quad \text{and} \quad u_\varphi \in L^p(a, b; D(M^3)). \quad (4.8)$$

Moreover, from (4.5) and (2.4), for $x > a$, we have

$$\begin{aligned} u'''_\psi(x) &= M^2u'_\psi(x) + kLe^{(x-a)L}\psi \\ &= M^2(L^2e^{(x-a)L} - M^2e^{(x-a)L})\psi + kLe^{(x-a)L}\psi \\ &= M^3(e^{(x-a)L} - e^{(x-a)L})\psi + M^2(L - M)e^{(x-a)L}\psi + kLe^{(x-a)L}\psi \\ &= M^3u_\psi(x) + kM^2(L+M)^{-1}e^{(x-a)L}\psi + kLe^{(x-a)L}\psi \\ &= M^3u_\psi(x) + k(M^2(L+M)^{-1} + L)e^{(x-a)L}\psi \\ &= M^2u_\varphi(x) + k(M^2(L+M)^{-1} + L)M^{-1}e^{(x-a)L}\varphi. \end{aligned}$$

Since $(M^2(L+M)^{-1} + L)M^{-1} \in \mathcal{L}(X)$, from (4.8) and (4.1), we deduce

$$u'''_\psi \in L^p(a, b; X). \quad (4.9)$$

Furthermore, for $x > a$, we have

$$\begin{aligned} u_\psi^{(4)}(x) &= M^2u''_\psi(x) + k(M^2(L+M)^{-1} + L)M^{-1}Le^{(x-a)L}\varphi \\ &= M^3u_\varphi(x) + kM^2(L+M)^{-1}e^{(x-a)L}\varphi + k(M^2(L+M)^{-1} + L)M^{-1}Le^{(x-a)L}\varphi \\ &= M^3u_\varphi(x) + k(M^2(L+M)^{-1}L^{-1} + M^2(L+M)^{-1}M^{-1} + LM^{-1})Le^{(x-a)L}\varphi. \end{aligned}$$

From Lemma 4.1, since $\varphi = M\psi \in (D(M), X)_{\frac{1}{p}, p} = (D(L), X)_{\frac{1}{p}, p}$, we obtain

$$x \mapsto Le^{(x-a)L}\varphi \in L^p(a, b; X).$$

Then, since $M^2(L+M)^{-1}L^{-1} + M^2(L+M)^{-1}M^{-1} + LM^{-1} \in \mathcal{L}(X)$ and from (4.8), we deduce

$$u_\psi^{(4)} \in L^p(a, b; X). \quad (4.10)$$

From statement 1. (see above), (4.8), (4.9) and (4.10), we deduce the result.

Second step. Assume that $u_\psi \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4))$. From (3.1), with $k = n = 4$ and $j = 2$, we have $u_\psi''(a) \in (D(M^4), X)_{\frac{1}{4p} + \frac{1}{2}, p}$, then from (3.2), we deduce $u_\psi''(a) \in (D(M), X)_{1 + \frac{1}{p}, p}$. From (4.5), for $x > a$, we have

$$M^{-2}u_\psi''(x) = u_\psi(x) + kM^{-2}e^{(x-a)L}\psi.$$

Passing to the limit when x tends to a , we obtain $u_\psi''(a) = k\psi$ and thus

$$\psi = k^{-1}u_\psi''(a) \in (D(M), X)_{1 + \frac{1}{p}, p}.$$

Finally, since $u_\psi(x) = M^{-1}u_\varphi(x)$, from (4.5), we obtain

$$\forall x > a, \quad M^2u_\psi''(x) = M^3u_\varphi(x) + kML^{-1}Le^{(x-a)L}\varphi.$$

Applying Lemma 4.1 with $\varphi := M\psi \in (D(M), X)_{\frac{1}{p}, p}$, the fact that $ML^{-1} \in \mathcal{L}(X)$ and (4.8), the previous equality yields $u_\psi'' \in L^p(a, b; D(M^2))$.

5 Proof of Theorem 2.5

Note that from Remark 3.1, in Theorem 2.5, we only have to prove the reverse implications. For each kind of boundary conditions, the proof of Theorem 2.5 is divided in two steps. First, from the representation formula obtained in section 2.3, we show uniqueness of a classical solution. Then, we state the existence of this solution. At this end, we establish the following technical result.

Lemma 5.1. *Let $V \in \mathcal{L}(X)$ such that $I + V$ is invertible in $\mathcal{L}(X)$. Then, there exists $W \in \mathcal{L}(X)$ such that*

$$(I + V)^{-1} = I - W,$$

and $W(X) \subset V(X)$. Moreover, if T is a linear operator in X such that $V(X) \subset D(T)$ and for any $\psi \in D(T)$, $TV\psi = VT\psi$, then

$$\forall \psi \in D(T), \quad WT\psi = TW\psi.$$

Proof. We set

$$W := V(I + V)^{-1} \in \mathcal{L}(X).$$

Then, obviously $(I + V)^{-1} = I - W$ and $W(X) \subset V(X) \subset D(T)$. Moreover, for $\psi \in D(T)$

$$(I + V)WT\psi = VT\psi = TV\psi = T(I + V)W\psi = (I + V)TW\psi,$$

thus $WT\psi = TW\psi$. □

5.1 Proof of statement 1. (Boundary conditions (BC2))

Assume $(H_1) \sim (H_4)$ and

$$\varphi_1, \varphi_2 \in (D(A), X)_{1 + \frac{1}{2} + \frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}. \quad (5.1)$$

First we prove the uniqueness of the solution of (1.4)-(BC2) by determining the constants K_i , $i = 1, 2, 3, 4$, of the representation formula (2.7). Then, we show that the formula obtained constitutes a classical solution.

First step: Uniqueness.

If u is a classical solution of (1.4)-(BC2), then from Proposition 2.3, u satisfies (2.7). In order to obtain a simplified model, we set

$$\alpha_1 := \frac{K_1 - K_2}{2}, \quad \alpha_2 := \frac{K_3 - K_4}{2}, \quad \alpha_3 := \frac{K_1 + K_2}{2} \quad \text{and} \quad \alpha_4 := \frac{K_3 + K_4}{2}. \quad (5.2)$$

Then for *a.e.* $x \in (a, b)$, u is given by

$$\begin{aligned} u(x) = & \left(e^{(x-a)M} - e^{(b-x)M} \right) \alpha_1 + \left(e^{(x-a)L} - e^{(b-x)L} \right) \alpha_2 + \left(e^{(x-a)M} + e^{(b-x)M} \right) \alpha_3 \\ & + \left(e^{(x-a)L} + e^{(b-x)L} \right) \alpha_4 + F_{0,f}(x). \end{aligned} \quad (5.3)$$

Since $F_{0,f}$ satisfies (3.7), the boundary conditions (BC2) applied to $M^{-2}u(x)$ for *a.e.* $x \in (a, b)$, imply the following relations

$$\begin{aligned} & \left(I + e^{(b-a)M} \right) M^{-1} \alpha_1 + \left(I + e^{(b-a)L} \right) LM^{-2} \alpha_2 \\ + & \left(I - e^{(b-a)M} \right) M^{-1} \alpha_3 + \left(I - e^{(b-a)L} \right) LM^{-2} \alpha_4 = M^{-2} \left(\varphi_1 - F'_{0,f}(a) \right), \end{aligned} \quad (5.4)$$

$$\begin{aligned} & \left(e^{(b-a)M} + I \right) M^{-1} \alpha_1 + \left(e^{(b-a)L} + I \right) LM^{-2} \alpha_2 \\ + & \left(e^{(b-a)M} - I \right) M^{-1} \alpha_3 + \left(e^{(b-a)L} - I \right) LM^{-2} \alpha_4 = M^{-2} \left(\varphi_2 - F'_{0,f}(b) \right), \end{aligned} \quad (5.5)$$

$$\left(I - e^{(b-a)L} \right) (L^2 - M^2) M^{-2} \alpha_2 + \left(I + e^{(b-a)L} \right) (L^2 - M^2) M^{-2} \alpha_4 = M^{-2} \varphi_3, \quad (5.6)$$

$$\left(e^{(b-a)L} - I \right) (L^2 - M^2) M^{-2} \alpha_2 + \left(e^{(b-a)L} + I \right) (L^2 - M^2) M^{-2} \alpha_4 = M^{-2} \varphi_4. \quad (5.7)$$

Note that we have considered $M^{-2}u(x)$ for *a.e.* $x \in (a, b)$, because we do not know if $\alpha_i \in D(M)$, $i = 1, 2, 3, 4$. Using (2.3) and summing (5.4) with (5.5) and (5.6) with (5.7), we obtain the abstract system

$$\left\{ \begin{array}{l} \left(I + e^{(b-a)M} \right) M^{-1} \alpha_1 + \left(I + e^{(b-a)L} \right) LM^{-2} \alpha_2 \\ + \left(I - e^{(b-a)M} \right) M^{-1} \alpha_3 + \left(I - e^{(b-a)L} \right) LM^{-2} \alpha_4 = M^{-2} \left(\varphi_1 - F'_{0,f}(a) \right) \\ 2 \left(I + e^{(b-a)M} \right) M^{-1} \alpha_1 + 2 \left(I + e^{(b-a)L} \right) LM^{-2} \alpha_2 = M^{-2} \left(\varphi_1 + \varphi_2 - F'_{0,f}(a) - F'_{0,f}(b) \right) \\ k \left(I - e^{(b-a)L} \right) M^{-2} \alpha_2 + k \left(I + e^{(b-a)L} \right) M^{-2} \alpha_4 = M^{-2} \varphi_3 \\ \qquad \qquad \qquad 2k \left(I + e^{(b-a)L} \right) M^{-2} \alpha_4 = M^{-2} \left(\varphi_3 + \varphi_4 \right). \end{array} \right.$$

This leads to the system

$$\left\{ \begin{array}{l} M \left(I - e^{(b-a)M} \right) M^{-2} \alpha_3 + L \left(I - e^{(b-a)L} \right) M^{-2} \alpha_4 = M^{-2} \tilde{\varphi}_2 \\ M \left(I + e^{(b-a)M} \right) M^{-2} \alpha_1 + L \left(I + e^{(b-a)L} \right) M^{-2} \alpha_2 = M^{-2} \tilde{\varphi}_1 \\ \qquad \qquad \qquad k \left(I - e^{(b-a)L} \right) \alpha_2 = \frac{\varphi_3 - \varphi_4}{2} \\ \qquad \qquad \qquad k \left(I + e^{(b-a)L} \right) \alpha_4 = \frac{\varphi_3 + \varphi_4}{2}, \end{array} \right.$$

where

$$\tilde{\varphi}_1 := \frac{\varphi_1 + \varphi_2 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 := \frac{\varphi_1 - \varphi_2 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \quad (5.8)$$

If T is the generator of a bounded analytic semigroup with $0 \in \rho(T)$, then, from [21], p. 70, for any $n \in \mathbb{N}$, we have:

$$\exists \delta > 0, \exists M > 1 : \left\| e^{n(b-a)T} \right\|_{\mathcal{L}(X)} \leq M e^{-n(b-a)\delta}. \quad (5.9)$$

Therefore, there exists $n_0 \in \mathbb{N} \setminus \{0\}$ such that $\|e^{n_0(b-a)T}\|_{\mathcal{L}(X)} < 1$. So $0 \in \rho\left(I \pm \left(e^{(b-a)T}\right)^{n_0}\right)$ which implies that $0 \in \rho\left(I \pm e^{(b-a)T}\right)$. Then, for $T := L$ or M , we deduce

$$\begin{cases} \alpha_1 &= \left(I + e^{(b-a)M}\right)^{-1} \left(M^{-1}\tilde{\varphi}_1 - LM^{-1}\left(I + e^{(b-a)L}\right)\alpha_2\right) \\ \alpha_2 &= k^{-1}\left(I - e^{(b-a)L}\right)^{-1} \left(\frac{\varphi_3 - \varphi_4}{2}\right) \\ \alpha_3 &= \left(I - e^{(b-a)M}\right)^{-1} \left(M^{-1}\tilde{\varphi}_2 - LM^{-1}\left(I - e^{(b-a)L}\right)\alpha_4\right) \\ \alpha_4 &= k^{-1}\left(I + e^{(b-a)L}\right)^{-1} \left(\frac{\varphi_3 + \varphi_4}{2}\right). \end{cases} \quad (5.10)$$

It follows that u is uniquely determined by (5.3), (5.8) and (5.10).

Second step: Existence.

Consider this uniquely determined u and set

$$\tilde{K}_1 := K_1 + K_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad \text{and} \quad \tilde{K}_2 := K_2 + K_4 = \alpha_3 + \alpha_4 - \alpha_1 - \alpha_2,$$

thus, for any $x \in (a, b)$,

$$\begin{aligned} u(x) &= e^{(x-a)M}\tilde{K}_1 + e^{(b-x)M}\tilde{K}_2 + \left(e^{(x-a)L} - e^{(x-a)M}\right)K_3 \\ &\quad + \left(e^{(b-x)L} - e^{(b-x)M}\right)K_4 + F_{0,f}(x). \end{aligned} \quad (5.11)$$

Then, from Remark 4.2 and Theorem 2.4, it suffices to show that $\tilde{K}_1, \tilde{K}_2 \in (D(M), X)_{3+\frac{1}{p}, p}$ and $K_3, K_4 \in (D(M), X)_{1+\frac{1}{p}, p}$. From Lemma 5.1, for $T := L$ or M , we have

$$\left(I \pm e^{2(b-a)T}\right)^{-1} = I + R_{\pm}, \quad \text{where } R_{\pm}(X) \subset D(M^{\infty}) \quad \text{and} \quad R_{\pm}T = TR_{\pm}.$$

Hence, from (5.10), there exists $R_i \in D(M^{\infty})$, $i = 1, 2, 3, 4$, such that

$$\begin{cases} \alpha_1 &= \frac{1}{2}M^{-1}(2\tilde{\varphi}_1 - k^{-1}L(\varphi_3 - \varphi_4)) + R_1 \\ \alpha_2 &= k^{-1}\frac{\varphi_3 - \varphi_4}{2} + R_2 \\ \alpha_3 &= \frac{1}{2}M^{-1}(2\tilde{\varphi}_2 - k^{-1}L(\varphi_3 + \varphi_4)) + R_3 \\ \alpha_4 &= k^{-1}\frac{\varphi_3 + \varphi_4}{2} + R_4. \end{cases}$$

From (5.2) and (5.8), we obtain

$$\begin{cases} K_1 &= M^{-1}(\varphi_1 - F'_{0,f}(a)) - k^{-1}LM^{-1}\varphi_3 + \tilde{R}_1 \\ K_2 &= -M^{-1}(\varphi_2 - F'_{0,f}(b)) - k^{-1}LM^{-1}\varphi_4 + \tilde{R}_2 \\ K_3 &= k^{-1}\varphi_3 + \tilde{R}_3 \\ K_4 &= k^{-1}\varphi_4 + \tilde{R}_4, \end{cases}$$

where $\tilde{R}_i \in D(M^{\infty})$, $i = 1, 2, 3, 4$. From (3.2), we have

$$(D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} = (D(M^2), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} = (D(M), X)_{2+\frac{1}{p}, p}, \quad (5.12)$$

and

$$(D(A), X)_{\frac{1}{2p}, p} = (X, D(M))_{1+1-\frac{1}{p}, p} = (D(M), X)_{1+\frac{1}{p}, p}. \quad (5.13)$$

Since $F'_{0,f}(a) \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}$, from (2.4), (5.1), (5.12) and (5.13), we have

$$\begin{aligned} \tilde{K}_1 &= M^{-1}(\varphi_1 - F'_{0,f}(a)) + k^{-1}(I - LM^{-1})\varphi_3 + \tilde{R}_1 + \tilde{R}_3 \\ &= M^{-1}(\varphi_1 - F'_{0,f}(a)) - (L + M)^{-1}M^{-1}\varphi_3 + \tilde{R}_1 + \tilde{R}_3 \in (D(M), X)_{3+\frac{1}{p}, p}, \end{aligned}$$

and, in the same way, $\tilde{K}_2 \in (D(M), X)_{3+\frac{1}{p}, p}$. Moreover,

$$K_3 = k^{-1}\varphi_3 + \tilde{R}_3 \in (D(M), X)_{1+\frac{1}{p}, p} \quad \text{and} \quad K_4 = k^{-1}\varphi_4 + \tilde{R}_4 \in (D(M), X)_{1+\frac{1}{p}, p},$$

which gives the result.

5.2 Proof of statement 2. (Boundary conditions (BC3))

In all the sequel, we assume $(H_1) \sim (H_5)$. For statements 1. and 2., the representation formula is easily obtained by taking into account the boundary conditions. But for statement 3. (and also statement 4.), to build the representation, we need the invertibility of some determinant operators.

More precisely, we have to prove that U and V given by

$$\begin{cases} U & := I - e^{(b-a)(L+M)} - k^{-1}(L+M)^2 \left(e^{(b-a)M} - e^{(b-a)L} \right) \\ V & := I - e^{(b-a)(L+M)} + k^{-1}(L+M)^2 \left(e^{(b-a)M} - e^{(b-a)L} \right), \end{cases} \quad (5.14)$$

are invertible with bounded inverse. This will be a consequence of a functional calculus result. To state this result, we need some notations and technical lemmas. For $\theta \in (0, \pi)$, we denote by $H(S_\theta)$ the space of holomorphic functions on S_θ (defined by (2.1)) with values in \mathbb{C} . Moreover, we consider the following subspace of $H(S_\theta)$:

$$\mathcal{E}_\infty(S_\theta) := \{f \in H(S_\theta) : f = O(|z|^{-s}) \text{ } (|z| \rightarrow +\infty) \text{ for some } s > 0\}.$$

In other words, $\mathcal{E}_\infty(S_\theta)$ is the space of polynomial decreasing holomorphic functions at ∞ . Let T be an invertible sectorial operator of angle $\theta_T \in (0, \pi)$. If $f \in \mathcal{E}_\infty(S_\theta)$, with $\theta \in (\theta_T, \pi)$, then we can define, by functional calculus, $f(T) \in \mathcal{L}(X)$ (see [14], p. 45).

Set, for $z \in \mathbb{C}$

$$f_k(z) := 1 - e^{-c(\sqrt{z} + \sqrt{z+k})} - k^{-1}(\sqrt{z} + \sqrt{z+k})^2 \left(e^{-c\sqrt{z}} - e^{-c\sqrt{z+k}} \right),$$

and

$$g_k(z) := 1 - e^{-c(\sqrt{z} + \sqrt{z+k})} + k^{-1}(\sqrt{z} + \sqrt{z+k})^2 \left(e^{-c\sqrt{z}} - e^{-c\sqrt{z+k}} \right),$$

where $c := b - a > 0$. Note that, we can write formally $U = f_k(-A)$ and $V = g_k(-A)$.

Lemma 5.2. *For any $k > 0$, f_k and g_k do not vanish in $\mathbb{R}^+ \setminus \{0\}$.*

Proof. First, we consider for $\tau > 0$, $\varphi(\tau) := \tau \frac{1 + e^{-\tau}}{1 - e^{-\tau}}$, then φ is a strictly increasing function on $\mathbb{R}^+ \setminus \{0\}$ since

$$\varphi'(\tau) = 2 \frac{\sinh(\tau) - \tau}{e^{-\tau} (e^\tau - 1)^2} > 0, \quad \tau > 0.$$

Let $x > 0$. Since $f_k(x) \leq g_k(x)$, it suffices to show that $f_k(x) > 0$. Set $s := c\sqrt{x}$ and $t := c\sqrt{x+k}$, then $0 < s < t$ and

$$\begin{aligned} f_k(x) &= 1 - e^{-(s+t)} - \frac{t+s}{t-s} (e^{-s} - e^{-t}) \\ &= \frac{1}{t-s} \left[(t-s) \left(1 - e^{-(s+t)} \right) - (t+s) (e^{-s} - e^{-t}) \right] \\ &= \frac{1}{t-s} \left[t(1 - e^{-s})(1 + e^{-t}) - s(1 - e^{-t})(1 + e^{-s}) \right] \\ &= \frac{(1 - e^{-s})(1 - e^{-t})}{t-s} [\varphi(t) - \varphi(s)] > 0, \end{aligned}$$

which leads to the result. \square

Lemma 5.3. *Let P be a sectorial invertible operator in X of angle θ , for all $\theta \in (0, \pi)$. Let $f \in H(S_\theta)$, for some $\theta \in (0, \pi)$, be such that*

$$(i) \quad 1 - f \in \mathcal{E}_\infty(S_\theta),$$

(ii) $f \neq 0$ on $\mathbb{R}^+ \setminus \{0\}$.

Then, $f(P) \in \mathcal{L}(X)$, is invertible with bounded inverse.

Proof. Since we have $\frac{1}{f} = 1 + h$, where $h := \frac{1-f}{f}$, it suffices to show that $h \in \mathcal{E}_\infty(S_{\theta_0})$, for some $\theta_0 \in (0, \pi)$, to obtain the result. Indeed, since $\sigma(P) \subset S_{\theta_0}$, from [14], pp. 28 and 45, in this case we can set:

$$\frac{1}{f}(P) = I + h(P) \in \mathcal{L}(X),$$

and the invertibility of $f(P)$ is obtained by writing

$$\frac{1}{f}(P)f(P) = \left(\frac{1}{f} \times f\right)(P) = 1(P) = I,$$

and similarly, $f(P)\frac{1}{f}(P) = I$.

From (i), there exists $R > 0$ such that, for all $z \in S_\theta$ with $|z| > R$, we have $|1 - f(z)| < 1/2$. Then $f(z)$ is never equal to 0 if $z \in S_\theta$ and $|z| > R$, so

$$h \in H\left(S_\theta \cap \left(\mathbb{C} \setminus \overline{B(0, R)}\right)\right). \quad (5.15)$$

Let $\theta_1 \in (0, \theta)$. In the compact set $\overline{S_{\theta_1}} \cap \overline{B(0, R)}$, the holomorphic function f has only a finite number of zeros, which from (ii), are not in $(0, R]$. So there exists $\theta_0 \in (0, \theta_1)$ such that $f \neq 0$ in $S_{\theta_0} \cap \overline{B(0, R)}$. Thus $h \in H\left(S_{\theta_0} \cap \overline{B(0, R)}\right)$ and from (5.15), we deduce $h \in H(S_{\theta_0})$.

Since $1 - f \in \mathcal{E}_\infty(S_{\theta_0})$, there exists $R_0 > 0$ such that, if $z \in S_{\theta_0}$ and $|z| > R_0$, then $|1 - f| < C|z|^{-\alpha}$ where $\alpha > 0$, $C > 0$.

Moreover, since for all $z \in S_{\theta_0}$ with $|z| > \max(R, R_0)$ we have $|1 - f(z)| < 1/2$, we deduce that $|f(z)| > 1/2$. So, for all $z \in S_{\theta_0}$ with $|z| > \max(R, R_0)$ we obtain $|h(z)| \leq 2C|z|^{-\alpha}$. It follows that $h \in \mathcal{E}_\infty(S_{\theta_0})$. \square

Proposition 5.4. U and V , defined by (5.14), are invertible with bounded inverse.

Proof. We first consider the case $k > 0$. Due to (H_5) , $P := -A$ satisfies the assumptions of Lemma 5.3. Moreover, it is clear that for a given $\theta \in (0, \pi)$, $f_k \in H(S_\theta)$ and $1 - f_k \in \mathcal{E}_\infty(S_\theta)$. Furthermore, from Lemma 5.2, f_k does not vanish on $\mathbb{R}^+ \setminus \{0\}$. So applying Lemma 5.3, we get that $U = f_k(-A)$ is invertible. The invertibility of $V = g_k(-A)$ is obtained in the same way.

Now, assume that $k < 0$. The linear operator $-A + kI$ is a sectorial operator from Remark 2.1 5. and as in the first case, setting $l := -k > 0$, we deduce that $U = f_l(-A + kI)$ and $V = g_l(-A + kI)$ are invertible with bounded inverse. \square

Now, we prove Theorem 2.5 for the boundary conditions (BC3). The proof is divided in two steps. First we prove the uniqueness of the solution of (1.4)-(BC3) by determining the constants K_i , $i = 1, 2, 3, 4$, of the representation formula (2.7). Then, we show that the formula obtained is a classical solution.

First step: Uniqueness.

Assume that

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (5.16)$$

If u is a classical solution of (1.4)-(BC3) then, as in section 5.1, u satisfies (5.3). Since $F_{0,f}$ satisfies (3.7), we apply the boundary conditions (BC3) to $M^{-1}u(\cdot)$ and obtain, by similar arguments as in section 5.1, the two following systems

$$\begin{cases} \left(I - e^{(b-a)M} \right) M^{-1}\alpha_1 + \left(I - e^{(b-a)L} \right) M^{-1}\alpha_2 = M^{-1} \left(\frac{\varphi_1 - \varphi_2}{2} \right) \\ M \left(I + e^{(b-a)M} \right) M^{-1}\alpha_1 + L \left(I + e^{(b-a)L} \right) M^{-1}\alpha_2 = M^{-1}\tilde{\varphi}_1, \end{cases} \quad (5.17)$$

and

$$\begin{cases} \left(I + e^{(b-a)M} \right) M^{-1} \alpha_3 + \left(I + e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \left(\frac{\varphi_1 + \varphi_2}{2} \right) \\ M \left(I - e^{(b-a)M} \right) M^{-1} \alpha_3 + L \left(I - e^{(b-a)L} \right) M^{-1} \alpha_4 = M^{-1} \tilde{\varphi}_2, \end{cases} \quad (5.18)$$

where

$$\tilde{\varphi}_1 := \frac{\varphi_3 + \varphi_4 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 := \frac{\varphi_3 - \varphi_4 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \quad (5.19)$$

Then, the unique determination of the constants α_i , $i = 1, 2, 3, 4$, is equivalent to show that the matrices Λ_1 and Λ_2 of the systems (5.17) and (5.18) have an invertible determinant. From (2.4), we have

$$\begin{aligned} \det(\Lambda_1) &= \begin{vmatrix} (I - e^{(b-a)M})M^{-1} & (I - e^{(b-a)L})M^{-1} \\ M(I + e^{(b-a)M})M^{-1} & L(I + e^{(b-a)L})M^{-1} \end{vmatrix} \\ &= \left((I - e^{(b-a)M})L(I + e^{(b-a)L}) - M(I + e^{(b-a)M})(I - e^{(b-a)L}) \right) M^{-2} \\ &= \left(L - M - (L - M)e^{(b-a)(L+M)} - (L + M)(e^{(b-a)M} - e^{(b-a)L}) \right) M^{-2} \\ &= kM^{-2}(L + M)^{-1}U, \end{aligned}$$

and in the same way

$$\begin{aligned} \det(\Lambda_2) &= \begin{vmatrix} (I + e^{(b-a)M})M^{-1} & (I + e^{(b-a)L})M^{-1} \\ M(I - e^{(b-a)M})M^{-1} & L(I - e^{(b-a)L})M^{-1} \end{vmatrix} \\ &= kM^{-2}(L + M)^{-1}V, \end{aligned}$$

where U and V are given by (5.14). Since U and V are invertible with bounded inverse (see Proposition 5.4), from (5.17) and (5.18), we deduce

$$\begin{cases} \alpha_1 = \frac{1}{2k}(L + M)U^{-1} \left[L(I + e^{(b-a)L})(\varphi_1 - \varphi_2) - 2(I - e^{(b-a)L})\tilde{\varphi}_1 \right] \\ \alpha_2 = -\frac{1}{2k}(L + M)U^{-1} \left[M(I + e^{(b-a)M})(\varphi_1 - \varphi_2) - 2(I - e^{(b-a)M})\tilde{\varphi}_1 \right] \\ \alpha_3 = \frac{1}{2k}(L + M)V^{-1} \left[L(I - e^{(b-a)L})(\varphi_1 + \varphi_2) - 2(I + e^{(b-a)L})\tilde{\varphi}_2 \right] \\ \alpha_4 = -\frac{1}{2k}(L + M)V^{-1} \left[M(I - e^{(b-a)M})(\varphi_1 + \varphi_2) - 2(I + e^{(b-a)M})\tilde{\varphi}_2 \right]. \end{cases} \quad (5.20)$$

Note that, since $F_{0,f} \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^2))$, from Remark 3.1 we have

$$F'_{0,f}(a) \in (D(M^2), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad F'_{0,f}(b) \in (D(M^2), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}. \quad (5.21)$$

This combined with (5.16) implies that $F'_{0,f}(a), F'_{0,f}(b), \varphi_i \in D(M^2)$, $i = 1, 2, 3, 4$. Thus all the previous equalities are well defined. This shows that the classical solution of (1.4)-(BC3) is characterized by (5.3), (5.19) and (5.20).

Second step: Existence.

It suffices to prove that u , determined in the first step, is a classical solution of (1.4)-(BC3). From Lemma 5.1, we have

$$U^{-1} = I + R_U \quad \text{and} \quad V^{-1} = I + R_V.$$

Set $T := L$ or M . Then, from (4.2), since $Te^{xT} = e^{xT}T$ on $D(T)$, for any $x \geq 0$, Lemma 5.1 implies

$$R_U(X), R_V(X) \subset D(M^\infty), \quad R_U T = T R_U \quad \text{and} \quad R_V T = T R_V.$$

This combined with (4.2) implies that there exist $R_i \in D(M^\infty)$, $i = 1, 2, 3, 4$, such that

$$\begin{cases} \alpha_1 &= \frac{1}{2k}(L+M)[L(\varphi_1 - \varphi_2) - 2\tilde{\varphi}_1] + R_1 \\ \alpha_2 &= -\frac{1}{2k}(L+M)[M(\varphi_1 - \varphi_2) - 2\tilde{\varphi}_1] + R_2 \\ \alpha_3 &= \frac{1}{2k}(L+M)[L(\varphi_1 + \varphi_2) - 2\tilde{\varphi}_2] + R_3 \\ \alpha_4 &= -\frac{1}{2k}(L+M)[M(\varphi_1 + \varphi_2) - 2\tilde{\varphi}_2] + R_4. \end{cases}$$

Then, from (5.2) and (5.19), we obtain

$$\begin{cases} K_1 &= \frac{1}{k}(L+M)[L\varphi_1 - \varphi_3 + F'_{0,f}(a)] + \tilde{R}_1 \\ K_2 &= \frac{1}{k}(L+M)[L\varphi_2 + \varphi_4 - F'_{0,f}(b)] + \tilde{R}_3 \\ K_3 &= -\frac{1}{k}(L+M)[M\varphi_1 - \varphi_3 + F'_{0,f}(a)] + \tilde{R}_2 \\ K_4 &= -\frac{1}{k}(L+M)[M\varphi_2 + \varphi_4 - F'_{0,f}(b)] + \tilde{R}_4, \end{cases} \quad (5.22)$$

with $\tilde{R}_i \in D(M^\infty)$, $i = 1, 2, 3, 4$. We set

$$\tilde{K}_1 := K_1 + K_3 \quad \text{and} \quad \tilde{K}_2 := K_2 + K_4,$$

thus, from (5.11), for any $x \in (a, b)$,

$$u(x) = e^{(x-a)M}\tilde{K}_1 + e^{(b-x)M}\tilde{K}_2 + \left(e^{(x-a)L} - e^{(x-a)M}\right)K_3 + \left(e^{(b-x)L} - e^{(b-x)M}\right)K_4 + F_{0,f}(x).$$

From (3.2), we have

$$(D(A), X)_{1+\frac{1}{2p}, p} = (D(M), X)_{3+\frac{1}{p}, p}. \quad (5.23)$$

Hence, since $\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2p}, p}$, from (2.3), we have

$$\tilde{K}_1 = K_1 + K_3 = \frac{1}{k}(L+M)(L-M)\varphi_1 + \tilde{R}_1 + \tilde{R}_3 = \varphi_1 + \tilde{R}_1 + \tilde{R}_3 \in (D(M), X)_{3+\frac{1}{p}, p},$$

and, in the same way,

$$\tilde{K}_2 = K_2 + K_4 = \varphi_2 + \tilde{R}_2 + \tilde{R}_4 \in (D(M), X)_{3+\frac{1}{p}, p}.$$

Then, from Lemma 4.1 and Remark 4.2, we obtain that

$$u_M : x \mapsto e^{(x-a)M}\tilde{K}_1 + e^{(b-x)M}\tilde{K}_2,$$

satisfies

$$u_M \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2)) \quad \text{and} \quad u''_M \in L^p(a, b; D(A)). \quad (5.24)$$

Thus, from (5.21), (5.16) and (5.23), $M\varphi_1, \varphi_3, F'_{0,f}(a) \in (D(M), X)_{2+\frac{1}{p}, p}$. From (5.22), we deduce

$$K_3 \in (D(M), X)_{1+\frac{1}{p}, p}.$$

So, from Theorem 2.4, we obtain that $v_{K_3} : x \mapsto (e^{(x-a)L} - e^{(x-a)M})K_3$ satisfies

$$v_{K_3} \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2)) \quad \text{and} \quad v''_{K_3} \in L^p(a, b; D(A)). \quad (5.25)$$

By the same arguments, $v_{K_4} : x \mapsto (e^{(x-a)L} - e^{(x-a)M})K_4$ satisfies

$$v_{K_4} \in W^{4,p}(a, b; X) \cap L^p(a, b; D(A^2)) \quad \text{and} \quad v''_{K_4} \in L^p(a, b; D(A)). \quad (5.26)$$

Since $F_{0,f}$ is a classical solution of (1.4), from (5.24), (5.25) and (5.26), we deduce that u is a classical solution of (1.4)-(BC3).

5.3 Proof of statement 3. (Boundary conditions (BC4))

We proceed as in the proof of the previous statement. We only point out the differences between the two proofs.

First step: Uniqueness.

Assume that

$$\varphi_1, \varphi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(A), X)_{\frac{1}{2p}, p}. \quad (5.27)$$

If u is a classical solution of (1.4)-(BC4), then u is given by (5.3). Since $F_{0,f}$ satisfies (3.7), the boundary conditions (BC4) applied to $M^{-1}u'(a), M^{-1}u'(b), M^{-2}u''(a), M^{-2}u''(b)$ and the same computations as in the proof of subsection 5.1 lead to the two second order systems

$$\begin{cases} (I + e^{(b-a)M})\alpha_1 + LM^{-1}(I + e^{(b-a)L})\alpha_2 = M^{-1}\tilde{\varphi}_1 \\ (I - e^{(b-a)M})\alpha_1 + L^2M^{-2}(I - e^{(b-a)L})\alpha_2 = M^{-2}\frac{\varphi_3 - \varphi_4}{2}, \end{cases} \quad (5.28)$$

and

$$\begin{cases} (I - e^{(b-a)M})\alpha_3 + LM^{-1}(I - e^{(b-a)L})\alpha_4 = M^{-1}\tilde{\varphi}_2 \\ (I + e^{(b-a)M})\alpha_3 + L^2M^{-2}(I + e^{(b-a)L})\alpha_4 = M^{-2}\frac{\varphi_3 + \varphi_4}{2}, \end{cases} \quad (5.29)$$

where

$$\tilde{\varphi}_1 := \frac{\varphi_1 + \varphi_2 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 := \frac{\varphi_1 - \varphi_2 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \quad (5.30)$$

Let Λ_1 and Λ_2 be their corresponding matrices. From (2.4), we have

$$\begin{aligned} \det(\Lambda_1) &= \begin{vmatrix} I + e^{(b-a)M} & LM^{-1}(I + e^{(b-a)L}) \\ I - e^{(b-a)M} & L^2M^{-2}(I - e^{(b-a)L}) \end{vmatrix} \\ &= L^2M^{-2}(I - e^{(b-a)L})(I + e^{(b-a)M}) - LM^{-1}(I + e^{(b-a)L})(I - e^{(b-a)M}) \\ &= kLM^{-2}V(L + M)^{-1}, \end{aligned}$$

and

$$\begin{aligned} \det(\Lambda_2) &= \begin{vmatrix} I - e^{(b-a)M} & LM^{-1}(I - e^{(b-a)L}) \\ I + e^{(b-a)M} & L^2M^{-2}(I + e^{(b-a)L}) \end{vmatrix} \\ &= L^2M^{-2}(I - e^{(b-a)M})(I + e^{(b-a)L}) - LM^{-1}(I - e^{(b-a)L})(I + e^{(b-a)M}) \\ &= kLM^{-2}(L + M)^{-1}U, \end{aligned}$$

where U and V are given by (5.14). Since U and V are invertible in $\mathcal{L}(X)$ (see Proposition 5.4), from (5.28) and (5.29), we deduce

$$\begin{cases} \alpha_1 = \frac{1}{2k}(L + M)V^{-1} \left[2(I - e^{(b-a)L})LM^{-1}\tilde{\varphi}_1 - (I + e^{(b-a)L})M^{-1}(\varphi_3 - \varphi_4) \right] \\ \alpha_2 = -\frac{1}{2k}(L + M)V^{-1} \left[2(I - e^{(b-a)M})ML^{-1}\tilde{\varphi}_1 - (I + e^{(b-a)M})L^{-1}(\varphi_3 - \varphi_4) \right] \\ \alpha_3 = \frac{1}{2k}(L + M)U^{-1} \left[2(I + e^{(b-a)L})LM^{-1}\tilde{\varphi}_2 - (I - e^{(b-a)L})M^{-1}(\varphi_3 + \varphi_4) \right] \\ \alpha_4 = -\frac{1}{2k}(L + M)U^{-1} \left[2(I + e^{(b-a)M})ML^{-1}\tilde{\varphi}_2 - (I - e^{(b-a)M})L^{-1}(\varphi_3 + \varphi_4) \right]. \end{cases} \quad (5.31)$$

Then, from (5.2) and (5.31) we obtain the uniqueness of the solution of (1.4)-(BC4).

Second step: Existence.

Let u , determined in the first step, then u is also given by (5.11). Then, it suffices to show that $\tilde{K}_1, \tilde{K}_2 \in (D(M), X)_{3+\frac{1}{p}, p}$ and $K_3, K_4 \in (D(M), X)_{1+\frac{1}{p}, p}$. By the same arguments as in the proof of subsection 5.2, there exist $R_i \in D(M^\infty)$, $i = 1, 2, 3, 4$, such that

$$\begin{cases} \alpha_1 &= \frac{1}{2k} (L + M) [2LM^{-1}\tilde{\varphi}_1 - M^{-1}(\varphi_3 - \varphi_4)] + R_1 \\ \alpha_2 &= -\frac{1}{2k} (L + M) [2ML^{-1}\tilde{\varphi}_1 - L^{-1}(\varphi_3 - \varphi_4)] + R_2 \\ \alpha_3 &= \frac{1}{2k} (L + M) [2LM^{-1}\tilde{\varphi}_2 - M^{-1}(\varphi_3 + \varphi_4)] + R_3 \\ \alpha_4 &= -\frac{1}{2k} (L + M) [2ML^{-1}\tilde{\varphi}_2 - L^{-1}(\varphi_3 + \varphi_4)] + R_4. \end{cases}$$

Then, from (5.2) and (5.30), we obtain

$$\begin{cases} K_1 &= \frac{1}{k} (L + M) [LM^{-1}(\varphi_1 - F'_{0,f}(a)) - M^{-1}\varphi_3] + \tilde{R}_1 \\ K_2 &= \frac{1}{k} (L + M) [LM^{-1}(-\varphi_2 + F'_{0,f}(b)) - M^{-1}\varphi_4] + \tilde{R}_2 \\ K_3 &= -\frac{1}{k} (L + M) [ML^{-1}(\varphi_1 - F'_{0,f}(a)) - L^{-1}\varphi_3] + \tilde{R}_3 \\ K_4 &= -\frac{1}{k} (L + M) [ML^{-1}(-\varphi_2 + F'_{0,f}(b)) - L^{-1}\varphi_4] + \tilde{R}_4, \end{cases} \quad (5.32)$$

with $\tilde{R}_i \in D(M^\infty)$, $i = 1, 2, 3, 4$. Then, combining (5.12) and (5.13), since $\varphi_1, F'_{0,f}(a) \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p}, p}$ and $\varphi_3 \in (D(A), X)_{\frac{1}{2p}, p}$, from (2.3), we have

$$\begin{aligned} \tilde{K}_1 &= K_1 + K_3 = \frac{1}{k} (L + M) [(LM^{-1} - ML^{-1})(\varphi_1 - F'_{0,f}(a)) + (L^{-1} - M^{-1})\varphi_3] + \tilde{R}_1 + \tilde{R}_2 \\ &= \frac{1}{k} (L + M) [(L^2 - M^2)M^{-1}L^{-1}(\varphi_1 - F'_{0,f}(a)) - (L - M)M^{-1}L^{-1}\varphi_3] + \tilde{R}_1 + \tilde{R}_2 \\ &= (L + M)M^{-1}L^{-1}(\varphi_1 - F'_{0,f}(a)) - M^{-1}L^{-1}\varphi_3 + \tilde{R}_1 + \tilde{R}_2 \in (D(M), X)_{3+\frac{1}{p}, p}, \end{aligned}$$

and, in the same way,

$$\tilde{K}_2 = K_2 + K_4 = (L + M)M^{-1}L^{-1}(F'_{0,f}(b) - \varphi_2) - M^{-1}L^{-1}\varphi_4 + \tilde{R}_2 + \tilde{R}_4 \in (D(M), X)_{3+\frac{1}{p}, p}.$$

Moreover, since $(L + M)\varphi_1, (L + M)F'_{0,f}(a), \varphi_3 \in (D(M), X)_{2+\frac{1}{p}, p}$, from (5.32), we deduce

$$K_3 \in (D(M), X)_{1+\frac{1}{p}, p},$$

and, in the same way, $K_4 \in (D(M), X)_{1+\frac{1}{p}, p}$, which leads to the result.

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