Generation of analytic semigroups for some generalized diffusion operators in L^p-spaces

Rabah Labbas, Stéphane Maingot & Alexandre Thorel

Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, ISCN, 76600 Le Havre, France.

rabah.labbas@univ-lehavre.fr, stephane.maingot@univ-lehavre.fr, alexandre.thorel@univ-lehavre.fr

Abstract

We consider some generalized diffusion operators of fourth order and their corresponding abstract Cauchy problem. Then, using semigroups techniques and functional calculus, we study the invertibility and the spectral properties of each operator. Therefore, we prove that we have generation of C_0-semigroup in each case. We also point out when these semigroups become analytic.

Key Words and Phrases: Fourth order boundary value problem, sectorial operators, spectral estimates, functional calculus, generation of analytic semigroups.

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1 Introduction

In this article, we study five abstract problems modelling a class of generalized diffusion problems set on a cylindrical space $\Omega = (a,b)\times\omega$ where ω is a bounded regular open set of $\mathbb{R}^{n-1}$, $n \geqslant 2$. By generalized diffusion problems, we mean a linear combination of the laplacian and the biharmonic operator. Here, the biharmonic term represents the long range diffusion, whereas the laplacian represents the short range diffusion.

This kind of problem arises in various concrete applications in physics, engineering and biology. For instance, in elasticity problems, we can cite [6], [19] or [32]. In electrostatic, we refer to [4], [17] or [23] and in plates theory, we refer to [12], [15] or [36]. In population dynamics, we also refer to [5], [21], [22], [29] or [28] and references therein cited.

Let $T > 0$, $k \in \mathbb{R}$ and $f \in L^p((0,T)\times\Omega)$, $p \in (1,+\infty)$. We consider, as an application model, the following generalized diffusion problem :

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x,y) = -\Delta^2 v(t,x,y) + k\Delta v(t,x,y) + f(t,x,y), & t\in(0,T],\ x\in(a,b),\ y\in\omega, \\ v(0,x,y) = v_0(x,y), & x\in(a,b),\ y\in\omega, \\ v(t,x,\zeta) = \Delta v(t,x,\zeta) = 0, & t\in(0,T],\ (x,\zeta)\in(a,b)\times\partial\omega \\ \text{Boundary Conditions (BC)} & \text{on } \{a,b\}\times\omega, \end{cases} \tag{1}$$

where v is a density, v_0 is given in a suitable space and the boundary conditions (BC) denote one of the following homogeneous boundary conditions:

$$\begin{cases} v(t,a,y) &=& 0, & v(t,b,y) &=& 0, & t\in(0,T],\ y\in\omega, \\ \partial_x^2 v(t,a,y) &=& 0, & \partial_x^2 v(t,b,y) &=& 0, & t\in(0,T],\ y\in\omega, \end{cases}$$

$$\begin{cases} \partial_x v(t,a,y) &=& 0, & \partial_x v(t,b,y) &=& 0, & t\in(0,T],\ y\in\omega, \\ \partial_x^2 v(t,a,y) + \Delta_y v(t,a,y) &=& 0, & \partial_x^2 v(t,b,y) + \Delta_y v(t,b,y) &=& 0, & t\in(0,T],\ y\in\omega, \end{cases}$$

$$\begin{cases} v(t,a,y) &=& 0, & v(t,b,y) &=& 0, & t\in(0,T],\ y\in\omega, \\ \partial_x v(t,a,y) &=& 0, & \partial_x v(t,b,y) &=& 0, & t\in(0,T],\ y\in\omega, \end{cases}$$

$$\begin{cases} \partial_x v(t,a,y) &= 0, & \partial_x v(t,b,y) &= 0, & t \in (0,T], \ y \in \omega, \\ \partial_x^2 v(t,a,y) &= 0, & \partial_x^2 v(t,b,y) &= 0, & t \in (0,T], \ y \in \omega, \end{cases}$$

or

$$\begin{cases} v(t,a,y) &= 0, & v(t,b,y) &= 0, & t \in (0,T], \ y \in \omega, \\ \partial_x^2 v(t,a,y) + \Delta_y v(t,a,y) &= 0, & \partial_x^2 v(b,y) + \Delta_y v(t,b,y) &= 0, & t \in (0,T], \ y \in \omega. \end{cases}$$

We set

$$\begin{cases} D(A_0) := W^{2,p}(\omega) \cap W_0^{1,p}(\omega) \\ \forall \psi \in D(A_0), \quad A_0 \psi = \Delta_y \psi. \end{cases}$$

Now, for $i = 1,2,3,4,5$, let us introduce the following linear operators which correspond to the abstract formulation of the spatial operator in (1):

$$\begin{cases} D(\mathcal{A}_{0,i}) &= \{u \in W^{4,p}(a,b;L^p(\omega)) \cap L^p(a,b;D(A_0^2)) \text{ and } u'' \in L^p(a,b;D(A_0)) : (\text{BCi})_0\} \\ \left[\mathcal{A}_{0,i} u\right](x) &= -u^{(4)}(x) - (2A_0 - kI)u''(x) - (A_0^2 - kA_0)u(x), \quad u \in D(\mathcal{A}_{0,i}), \ x \in (a,b). \end{cases}$$

Here, $(\text{BCi})_0$, $i = 1,2,3,4,5$, represents the following boundary conditions (BCi) in the homogeneous case:

$$\begin{cases} u(a) &= \varphi_1, & u(b) &= \varphi_2, \\ u''(a) &= \varphi_3, & u''(b) &= \varphi_4, \end{cases} \tag{BC1}$$

$$\begin{cases} u'(a) &= \varphi_1, & u'(b) &= \varphi_2, \\ u''(a) + A_0 u(a) &= \varphi_3, & u''(b) + A_0 u(b) &= \varphi_4, \end{cases} \tag{BC2}$$

$$\begin{cases} u(a) &= \varphi_1, & u(b) &= \varphi_2, \\ u'(a) &= \varphi_3, & u'(b) &= \varphi_4, \end{cases} \tag{BC3}$$

$$\begin{cases} u'(a) &= \varphi_1, & u'(b) &= \varphi_2, \\ u''(a) &= \varphi_3, & u''(b) &= \varphi_4, \end{cases} \tag{BC4}$$

or

$$\begin{cases} u(a) &= \varphi_1, & u(b) &= \varphi_2, \\ u''(a) + A_0 u(a) &= \varphi_3, & u''(b) + A_0 u(b) &= \varphi_4, \end{cases} \tag{BC5}$$

where $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in L^p(\omega)$.

In this work, we study operators $\mathcal{A}_i$, $i = 1,2,3,4,5$, wherein we have considered a more general operator A, satisfying some elliptic assumptions described in section 4, instead of A_0. Then, we study the spectral equation

$$(-\mathcal{A}_i - \lambda I)\, u = g,$$

where $g \in L^p(a,b;X)$ with $p \in (1,+\infty)$ and X a complex Banach space. This leads us to solve the abstract Cauchy problem

$$\begin{cases} v'(t) - \mathcal{A}_i v(t) = f(t), \quad t \in (0,T] \\ v(0) = v_0, \end{cases} \tag{2}$$

in two cases:

1. $f : [0,T] \longrightarrow L^p(a,b;X)$ with $p \in (1,+\infty)$ and v_0 is in suitable spaces,
2. $f : [0,T] \longrightarrow C^\theta([a,b];X)$ with $\theta \in (0,1)$, $v_0 \in D(\mathcal{A}_i)$ and f, v_0 satisfying some compatibility condition which will be specified in section 5.

Moreover, we will study, among others, the optimal regularity of the two following functions

$$u_\psi(x) = \left(e^{(x-a)\Lambda_2} - e^{(x-a)\Lambda_1}\right)\psi \quad \text{and} \quad v_\psi(x) = \left(e^{(b-x)\Lambda_2} - e^{(b-x)\Lambda_1}\right)\psi,$$

where $\psi \in X$, $x \in (a,b) \subset \mathbb{R}$, with $a < b$, $\Lambda_2 = \Lambda_1 + \mathcal{B}$; here Λ_1 is a boundedly invertible operator satisfying the Maximal Regularity property $(\mathcal{MR})$, see section 3.2.3, $\mathcal{B} \in \mathcal{L}(X)$ and $\Lambda_1\mathcal{B} = \mathcal{B}\Lambda_1$ on $D(\Lambda_1)$.

This optimal regularity, which is an interesting result in itself, will be very useful for the study of the spectral properties of $\mathcal{A}_i$.

This article is organized as follows. In section 2, we recall some classical definitions and results about sectorial operators and interpolation spaces. In section 3, we study a class of general fourth order linear abstract problem where we show the existence and uniqueness of the classical solution for each problem. In section 4, using the results of the previous section, we study the spectral properties of each $\mathcal{A}_i$ for $i = 1,2,3,4,5$ and we prove that $-\mathcal{A}_i$ is sectorial. We then obtain that $\mathcal{A}_i$ generates a C_0-semigroup which becomes analytic for some A as A_0. Finally, section 5 is devoted to an application focused on the study of problem (1).

2 Definitions and prerequisites

2.1 The class of Bounded Imaginary Powers of operators

Definition 2.1. A Banach space X is a UMD space if and only if for all $p \in (1,+\infty)$, the Hilbert transform is bounded from $L^p(\mathbb{R}, X)$ into itself (see [2] and [3]).

Definition 2.2. Let $\alpha \in (0,\pi)$. Sect(α) denotes the space of closed linear operators T_1 which satisfying

$$\begin{array}{ll} i) & \sigma(T_1) \subset \overline{S_\alpha}, \\ ii) & \forall\, \alpha' \in (\alpha,\pi), \quad \sup\left\{\|\lambda(\lambda I - T_1)^{-1}\|_{\mathcal{L}(X)} :\ \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}}\right\} < +\infty, \end{array}$$

where

$$S_\alpha := \left\{ \begin{array}{ll} \{z \in \mathbb{C} : z \neq 0 \ \text{ and } \ |\arg(z)| < \alpha\} & \text{if} \quad \alpha \in (0,\pi], \\ (0,+\infty) & \text{if} \quad \alpha = 0, \end{array} \right. \tag{3}$$

see [18], p. 19. Such an operator T_1 is called sectorial operator of angle α.

Remark 2.3. From [20], p. 342, we know that any injective sectorial operator T_1 admits imaginary powers T_1^{is}, $s \in \mathbb{R}$, but, in general, T_1^{is} is not bounded.

Definition 2.4. Let $\theta \in [0,\pi)$. We denote by BIP(X,θ), the class of sectorial injective operators T_2 such that

$$\begin{array}{ll} i) & \overline{D(T_2)} = \overline{R(T_2)} = X, \\ ii) & \forall\, s \in \mathbb{R}, \quad T_2^{is} \in \mathcal{L}(X), \\ iii) & \exists\, C \geq 1,\ \forall\, s \in \mathbb{R}, \quad ||T_2^{is}||_{\mathcal{L}(X)} \leq Ce^{|s|\theta}, \end{array}$$

see [31], p. 430.

2.2 Interpolation spaces

Here we recall some properties about real interpolation spaces in particular cases.

Definition 2.5. Let $T_3 : D(T_3) \subset X \longrightarrow X$ be a linear operator such that

$$(0, +\infty) \subset \rho(T_3) \quad \text{and} \quad \exists\, C > 0 : \forall\, t > 0, \quad \|t(T_3 - tI)^{-1}\|_{\mathcal{L}(X)} \leqslant C. \tag{4}$$

Let $m \in \mathbb{N} \setminus \{0\}$, $\theta \in (0,1)$ and $q \in [1, +\infty]$. We will use the real interpolation spaces

$$(D(T_3^m), X)_{\theta,q} = (X, D(T_3^m))_{1\text{-}\theta,q},$$

defined, for instance, [24] or [25].

In particular, for $m = 1$, we have the following characterization

$$(D(T_3), X)_{\theta,q} := \left\{\psi \in X : t \longmapsto t^{1-\theta}\|T_3(T_3 - tI)^{-1}\psi\|_X \in L^q_*(0, +\infty)\right\},$$

where $L^q_*(0, +\infty)$ is given by

$$L^q_*(0, +\infty; \mathbb{C}) := \left\{f \in L^q(0, +\infty) : \left(\int_0^{+\infty} |f(t)|^q \frac{dt}{t}\right)^{1/q} < +\infty\right\}, \quad \text{for } q \in [1, +\infty),$$

and for $q = +\infty$, by

$$L^\infty_*(0, +\infty; \mathbb{C}) := \left\{f \text{ measurable on } (0, +\infty) : \operatorname*{ess\,sup}_{t\in(0,+\infty)} |f(t)| < +\infty\right\},$$

see [7] p. 325, or [16], p. 665, Teorema 3, or section 1.14 of [37], where this space is denoted by $(X, D(T_3))_{1\text{-}\theta,q}$. Note that we can also characterize the space $(D(T_3), X)_{\theta,q}$ taking into account the Osservazione, p. 666, in [16].

We set also, for any $m \in \mathbb{N} \setminus \{0\}$

$$(D(T_3), X)_{m+\theta,q} \;:=\; \{\psi \in D(T_3^m) : T_3^m\psi \in (D(T_3), X)_{\theta,q}\},$$

and

$$(X, D(T_3))_{m+\theta,q} \;:=\; \{\psi \in D(T_3^m) : T_3^m\psi \in (X, D(T_3))_{\theta,q}\},$$

see [26], definition 3.2, p. 64.

Remark 2.6. Note that for T_3 satisfying (4), T_3^m is closed for any $m \in \mathbb{N} \setminus \{0\}$ since $\rho(T_3) \neq \emptyset$; consequently, if $m\theta < 1$, we have

$$(D(T_3^m), X)_{\theta,q} = (X, D(T_3^m))_{1-\theta,q} = (X, D(T_3))_{m-m\theta,q} = (D(T_3), X)_{(m-1)+m\theta,q} \subset D(T_3^{m-1}).$$

For more details see [25], (2.1.13), p. 43; [26] Proposition 3.8, p. 69 or [16], p. 676, Teorema 6.

2.3 Prerequisites

In this section, we recall some well-known facts, useful in our proofs.

Lemma 2.7 ([16]). Let T_3 be a linear operator satisfying (4). Let u such that

$$u \in W^{n,p}(a, b; X) \cap L^p(a, b; D(T_3^m)),$$

where $a, b \in \mathbb{R}$ with $a < b$, $n, m \in \mathbb{N} \setminus \{0\}$ and $p \in (1, +\infty)$. Then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $s \in \{a, b\}$, we have

$$u^{(j)}(s) \in (D(T_3^m), X)_{\frac{j}{n}+\frac{1}{np},p}.$$

This result is proved in [16], Teorema 2', p. 678.

Lemma 2.8. Let $\psi \in X$ and T_3 be a generator of a bounded analytic semigroup in X with $0 \in \rho(T_3)$. Then, for any $m \in \mathbb{N} \setminus \{0\}$ and $p \in [1, +\infty]$, the next properties are equivalent:

1. $x \mapsto T_3^m e^{(x-a)T_3}\psi \in L^p(a, +\infty; X)$
2. $\psi \in (D(T_3), X)_{m-1+\frac{1}{p},p}$
3. $x \mapsto e^{(x-a)T_3}\psi \in W^{m,p}(a, b; X)$
4. $x \mapsto T_3^m e^{(x-a)T_3}\psi \in L^p(a, b; X)$.

The equivalence between 1 and 2 is proved in [37]. The others are proved in [35], Lemma 3.2, p. 638-639.

Lemma 2.9 ([21])**.** Let $V \in \mathcal{L}(X)$ such that $0 \in \rho(I + V)$. Then

$$(I + V)^{-1} = I - V(I + V)^{-1},$$

and $V(I + V)^{-1}(X) \subset V(X)$. Moreover, if T is a linear operator in X such that $V(X) \subset D(T)$ and for $\psi \in D(T)$, $TV\psi = VT\psi$, then

$$\forall\, \psi \in D(T), \quad V(I + V)^{-1}T\psi = TV(I + V)^{-1}\psi.$$

This result is proved in [21], Lemma 5.1, p. 365.

3 General fourth order abstract problem

In this section, we study the following more general linear abstract equation of fourth order:

$$u^{(4)}(x) + (P + Q)u''(x) + PQu(x) = f(x), \quad x \in (a, b). \tag{5}$$

Here, P and Q are two linear operators on a Banach space X verifying some assumptions (see below) and $f \in L^p(a, b; X)$, with $p \in (1, +\infty)$.

We search a classical solution u of (5), which is a solution u of (5) such that

$$u \in W^{4,p}(a, b; X) \cap L^p(a, b; D(PQ)) \quad \text{and} \quad u'' \in L^p(a, b; D(P) \cap D(Q)).$$

Moreover, we consider, in this section, the boundary conditions (BCi), $i = 1, 2, 3, 4, 5$, where A_0 is replaced by P. We say that u is a classical solution of (5)-(BCi), $i = 1, 2, 3, 4, 5$, if u is a classical solution of (5) satisfying (BCi).

This study will allow us, in particular, to deduce all the spectral properties for operators $\mathcal{A}_i$, for each $i = 1, 2, 3, 4, 5$.

3.1 Assumptions, main results and some remarks

3.1.1 Assumptions

We assume the following hypotheses:

(H_1) X is a UMD space,

(H_2) P and Q are closed and $0 \in \rho(P) \cap \rho(Q)$,

(H_3) $D(P) = D(Q)$ and $P^{-1}Q^{-1} = Q^{-1}P^{-1}$,

(H_4) $-P, -Q \in \mathrm{BIP}(X, \theta_0)$, for $\theta_0 \in [0, \pi)$,

(H_5) $P - Q$ admits an invertible extension $B \in \mathcal{L}(X)$.

Some of our results will need a supplementary hypothesis:

$$(H_6) \quad 0 \in \rho(U) \cap \rho(V),$$

where

$$\begin{cases} U & := & I - e^{c(L+M)} - B^{-1}(L+M)^2\left(e^{cM} - e^{cL}\right) & = & I - T^- \in \mathcal{L}(X) \\ V & := & I - e^{c(L+M)} + B^{-1}(L+M)^2\left(e^{cM} - e^{cL}\right) & = & I - T^+ \in \mathcal{L}(X), \end{cases} \tag{6}$$

with $c := b - a > 0$, $L := -\sqrt{-Q}$ and $M := -\sqrt{-P}$.

Note that, from (H_4), $-P$ and $-Q$ are sectorial operators; this allows us to define

$$L = -\sqrt{-Q} \quad \text{and} \quad M = -\sqrt{-P}, \tag{7}$$

which generate bounded analytic semigroups on X and due to (H_3), $L + M$ also generates a bounded analytic semigroup on X, see for instance [31] Theorem 5, p. 443.

3.1.2 Main results

Theorem 3.1. Let $f \in L^p(a, b; X)$, with $p \in (1, +\infty)$. Assume that (H_1), (H_2), (H_3), (H_4) and (H_5) hold. Then,

1. there exists a unique classical solution u_1 of problem (5)-(BC1) if and only if

$$\varphi_1, \varphi_2 \in (D(P), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(P), X)_{\frac{1}{2p},p}. \tag{8}$$

This solution, denoted by $F_{\Phi,f}$ with $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, is given, for all $x \in [a, b]$, by

$$\begin{aligned} F_{\Phi,f}(x) &= \left(e^{(x-a)M} - e^{(b-x)M}e^{cM}\right) Z\varphi_1 + \left(e^{(b-x)M} - e^{(x-a)M}e^{cM}\right) Z\varphi_2 \\ &\quad + \frac{1}{2}\left(e^{(b-x)M}e^{cM} - e^{(x-a)M}\right) ZM^{-1}\int_a^b e^{(s-a)M}v_0(s)\, ds \\ &\quad + \frac{1}{2}\left(e^{(x-a)M}e^{cM} - e^{(b-x)M}\right) ZM^{-1}\int_a^b e^{(b-s)M}v_0(s)\, ds \\ &\quad + \frac{1}{2}M^{-1}\int_a^x e^{(x-s)M}v_0(s)\, ds + \frac{1}{2}M^{-1}\int_x^b e^{(s-x)M}v_0(s)\, ds, \end{aligned} \tag{9}$$

where

$$\begin{aligned} v_0(x) &:= \left(e^{(x-a)L} - e^{(b-x)L}e^{cL}\right) W(\varphi_3 + P\varphi_1) \\ &\quad + \left(e^{(b-x)L} - e^{(x-a)L}e^{cL}\right) W(\varphi_4 + P\varphi_2) \\ &\quad + \frac{1}{2}\left(e^{(b-x)L}e^{cL} - e^{(x-a)L}\right) WL^{-1}\int_a^b e^{(s-a)L}f(s)\, ds \\ &\quad + \frac{1}{2}\left(e^{(x-a)L}e^{cL} - e^{(b-x)L}\right) WL^{-1}\int_a^b e^{(b-s)L}f(s)\, ds \\ &\quad + \frac{1}{2}L^{-1}\int_a^x e^{(x-s)L}f(s)\, ds + \frac{1}{2}L^{-1}\int_x^b e^{(s-x)L}f(s)\, ds, \end{aligned} \tag{10}$$

with $Z := \left(I - e^{2cM}\right)^{-1}$ and $W := \left(I - e^{2cL}\right)^{-1}$.

Note that the existence of Z and W is ensured by [25], Proposition 2.3.6, p. 60.

2. there exists a unique classical solution u_5 of problem (5)-(BC5) if and only if

$$\varphi_1, \varphi_2 \in (D(P), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(P), X)_{\frac{1}{2p},p}. \tag{11}$$

In this case, the unique solution is $u_5 = F_{(\varphi_1,\varphi_2,\varphi_3-P\varphi_1,\varphi_4-P\varphi_2),f}$.

Theorem 3.2. Let $f \in L^p(a,b;X)$ with $p \in (1,+\infty)$. Assume that (H_1), (H_2), (H_3), (H_4) and (H_5) hold. Then

1. there exists a unique classical solution u_2 of (5)-(BC2) if and only if
$$\varphi_1, \varphi_2 \in (D(P),X)_{1+\frac{1}{2}+\frac{1}{2p},p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(P),X)_{\frac{1}{2p},p}. \tag{12}$$

If moreover, (H_6) holds, then

2. there exists a unique classical solution u_3 of (5)-(BC3) if and only if
$$\varphi_1, \varphi_2 \in (D(P),X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(P),X)_{1+\frac{1}{2}+\frac{1}{2p},p}, \tag{13}$$

3. there exists a unique classical solution u_4 of (5)-(BC4) if and only if
$$\varphi_1, \varphi_2 \in (D(P),X)_{1+\frac{1}{2}+\frac{1}{2p},p} \quad \text{and} \quad \varphi_3, \varphi_4 \in (D(P),X)_{\frac{1}{2p},p}. \tag{14}$$

3.1.3 Some remarks

Remark 3.3.

1. In [21], problems
$$\begin{cases} u^{(4)}(x) + (2A - kI)u''(x) + (A^2 - kA)u(x) = f(x), \quad x \in (a,b), \\ \text{(BCi)}, \end{cases}$$
for each $i = 1,2,3,4,5$, correspond to problems (5)-(BCi) where $P = A$ and $Q = A - kI$, with $k \in \mathbb{R}$ such that $[k,+\infty) \subset \rho(A)$.

2. Assumptions (H_2) and (H_3) involve: $D(PQ) = D(QP) = D(P^2) = D(Q^2)$. Moreover,
$$D(LM) = D(ML) = D(L^2) = D(M^2) \quad \text{and} \quad LM = ML. \tag{15}$$

3. Due to (H_4) and [18], Proposition 3.2.1, e), p. 71, we have
$$-L, -M \in \text{ BIP}\,(X, \theta_0/2);$$
so from [31], Theorem 4, p. 441, $-(L+M) \in \text{BIP}\,(X, \theta_0/2 + \varepsilon)$ with $\varepsilon > 0$. Moreover, $L + M$ is invertible with bounded inverse.

4. Assumption (H_5) means that operators $-P$ and $-Q$ satisfy the following property:
$$\exists B \in \mathcal{L}(X) \; : \; 0 \in \rho(B) \quad \text{and} \quad P = Q + B.$$
When $P = A$ and $Q = A - \mu I$ with $\mu \in \mathbb{C} \setminus \{0\}$, then $P - Q = \mu I \in \mathcal{L}(X)$ is invertible.

5. Assume that (H_1), (H_2), (H_3), (H_4) and (H_5) are satisfied.

   Then, (H_6) holds if $c = b - a$ is enough large. Indeed, since L, M and $L + M$ are invertible with bounded inverse and generate bounded analytic semigroups, there exist $\delta > 0$ and $C \geqslant 1$ (see [25], (2.1.1) and (2.1.2) p. 35 taking $\omega = -\delta$ where $\delta > 0$ is enough small) such that
$$\max\left(\left\|e^{c(L+M)}\right\|_{\mathcal{L}(X)}, \left\|M^2 e^{cM}\right\|_{\mathcal{L}(X)}, \left\|L^2 e^{cL}\right\|_{\mathcal{L}(X)}\right) \leqslant Ce^{-\delta c},$$

thus

$$
\begin{aligned}
M_B &:= \max\left(\|T^-\|_{\mathcal{L}(X)}, \left\|T^+\right\|_{\mathcal{L}(X)}\right) \\
&\leqslant \quad \left\|e^{c(L+M)}\right\|_{\mathcal{L}(X)} + \left\|B^{-1}(L+M)^2 M^{-2}\right\|_{\mathcal{L}(X)} \left\|M^2 e^{cM}\right\|_{\mathcal{L}(X)} \\
&\quad + \left\|B^{-1}(L+M)^2 L^{-2}\right\|_{\mathcal{L}(X)} \left\|L^2 e^{cL}\right\|_{\mathcal{L}(X)} \\
&\leqslant \left(1 + \left\|B^{-1}(L+M)^2 M^{-2}\right\|_{\mathcal{L}(X)} + \left\|B^{-1}(L+M)^2 L^{-2}\right\|_{\mathcal{L}(X)}\right) C e^{-\delta c}.
\end{aligned}
$$

Finally, for $c = b - a$ enough large, $\|T^-\|_{\mathcal{L}(X)} < 1$ and $\|T^+\|_{\mathcal{L}(X)} < 1$. It follows that U and V are invertible.

6. In some particular cases, we could check assumption (H_6) using functional calculus for sectorial operators, see for instance [21] where $B = k \in \mathbb{R} \setminus \{0\}$.

Remark 3.4.

1. Note that $P - Q \subset B \in \mathcal{L}(X)$ involves that $P - Q$ is closable but, in general, is not closed. In fact, if $P - Q$ is closed, then due to $\overline{D(P-Q)} = X$ and $P - Q \subset B \in \mathcal{L}(X)$, we deduce that $D(P-Q) = X$, thus $D(P) = D(Q) = X$. Then, $P - Q$ is closed only if $P - Q \in \mathcal{L}(X)$.

2. Since $L = -\sqrt{-Q}$ and $M = -\sqrt{-P}$ we have $P - Q = L^2 - M^2$. Moreover
$$(L-M)(L+M) = L^2 - M^2, \tag{16}$$
indeed, it is well known that if C_j, $j = 1,2,3,4$, are linear operators, then
$$C_1C_3 + C_1C_4 + C_2C_3 + C_2C_4 \subset (C_1 + C_2)(C_3 + C_4),$$
but not necessary the equality. In our case, from (15), we have
$$L^2 - M^2 = L^2 + LM - ML - M^2 \subset (L - M)(L + M),$$
and also $D(L^2 - M^2) = D(M^2)$. To conclude it suffices to check that
$$D((L-M)(L+M)) \subset D(L^2 - M^2) = D\left(M^2\right).$$
To this end, consider $\psi \in D((L-M)(L+M))$. Then $\psi \in D(L+M)$ and
$$(L+M)\psi \in D(L-M) = D(M).$$
Thus, there exists $\chi \in X$ such that $(L+M)\psi = M^{-1}\chi$. Hence $M\psi = (L+M)^{-1}\chi \in D(M)$, that is $\psi \in D(M^2) = D(L^2)$.

3. Recall that $L^2 - M^2 \subset B$ means that
$$\forall\, \psi \in D(M^2), \quad (L^2 - M^2)\psi = B\psi.$$
Hence
$$\forall\, \psi \in D(M), \quad (L-M)\psi = (P-Q)(L+M)^{-1}\psi = B(L+M)^{-1}\psi. \tag{17}$$

4. For $z \in \rho(-P) \cap \rho(-Q)$, we have
$$
\begin{aligned}
B(-Q - zI)^{-1}(-P - zI)^{-1} &= (P - Q)(-Q - zI)^{-1}(-P - zI)^{-1} \\
&= (-Q - zI + P + zI)(-Q - zI)^{-1}(-P - zI)^{-1} \\
&= (-P - zI)^{-1} - (-Q - zI)^{-1}.
\end{aligned}
$$
Thus
$$B(-Q - zI)^{-1}(-P - zI)^{-1} = (-P - zI)^{-1} - (-Q - zI)^{-1}. \tag{18}$$

3.2 Preliminary results

3.2.1 Particular solutions : proof of Theorem 3.1

1. The proof is similar to the one of Theorem 2.2, p. 355 in [21], thus we omit it.

2. It suffices to remark that condition (BC5) writes as

$$\left\{ \begin{array}{rclrcl} u(a) & = & \varphi_1, & u(b) & = & \varphi_2, \\ u''(a) & = & \varphi_3 - P\varphi_1, & u''(b) & = & \varphi_4 - P\varphi_2, \end{array} \right.$$

and then to apply the first statement.

3.2.2 Representation formula

We begin by two technical lemmas.

Lemma 3.5. Assume that (H_1), (H_2), (H_3), (H_4) and (H_5) hold. Then, $P^{-1}B = BP^{-1}$, which means that

$$\forall\ \psi \in D(P), \quad B\psi \in D(P) \quad \text{and} \quad PB\psi = BP\psi. \tag{19}$$

In the same way, we have $Q^{-1}B = BQ^{-1}$, which means that

$$\forall\ \psi \in D(Q), \quad B\psi \in D(Q) \quad \text{and} \quad QB\psi = BQ\psi. \tag{20}$$

Proof. First of all, note that from (H_3), $P - Q$ is resolvent commuting with P, Q, $P + Q$ and all linear combination of P and Q.

Let $\chi \in X$. There exists $(\chi_n)_{n\geqslant 0} \subset D(P)$ such that $\chi_n \underset{n\longrightarrow+\infty}{\longrightarrow} \chi$. Then, since we have $P^{-1}B, BP^{-1} \in \mathcal{L}(X)$ and $B = P - Q$ on $D(P)$, we obtain

$$\begin{array}{rclclcl} P^{-1}B\chi & = & \lim\limits_{n\longrightarrow+\infty} P^{-1}B\chi_n & = & \lim\limits_{n\longrightarrow+\infty} P^{-1}(P-Q)\chi_n & = & \lim\limits_{n\longrightarrow+\infty} (P-Q)P^{-1}\chi_n \\ & = & \lim\limits_{n\longrightarrow+\infty} BP^{-1}\chi_n & = & BP^{-1}\chi. & & \end{array}$$

Thus, $P^{-1}B = BP^{-1}$.

If $\psi \in D(P)$ then, setting $\chi = P\psi$, we have

$$B\psi = BP^{-1}\chi = P^{-1}B\chi = P^{-1}BP\psi.$$

Then $B\psi = P^{-1}BP\psi \in D(P)$ and $PB\psi = BP\psi$. In the same way, replacing P by Q in the previous proof, we obtain (20). □

Lemma 3.6. Assume that (H_1), (H_2), (H_3), (H_4) and (H_5) hold. Then, $D(L) = D(M)$.

Proof. By definition, using the Dunford-Riesz integral, we have

$$\left\{ \begin{array}{rcl} -M = \sqrt{-P} & = & \dfrac{1}{2i\pi}(I-P)\displaystyle\int_\gamma \frac{\sqrt{z}}{1+z}(-P-zI)^{-1}\ dz \\ -L = \sqrt{-Q} & = & \dfrac{1}{2i\pi}(I-Q)\displaystyle\int_\gamma \frac{\sqrt{z}}{1+z}(-Q-zI)^{-1}\ dz, \end{array} \right.$$

where γ is a sectorial curve surrounding $\sigma(-P) \cup \sigma(-Q)$, see [18], p. 61. Moreover,

$$\left\{ \begin{array}{l} \displaystyle\int_\gamma \frac{\sqrt{z}}{1+z}(-P-zI)^{-1}\ dz \in \mathcal{L}(X) \\ \displaystyle\int_\gamma \frac{\sqrt{z}}{1+z}(-Q-zI)^{-1}\ dz \in \mathcal{L}(X), \end{array} \right.$$

and

$$\begin{cases} \psi \in D(M) = D(\sqrt{-P}) & \iff \displaystyle\int_\gamma \frac{\sqrt{z}}{1+z}\,(-P-zI)^{-1}\,\psi\;dz \in D\,(P) \\ \psi \in D(L) = D(\sqrt{-Q}) & \iff \displaystyle\int_\gamma \frac{\sqrt{z}}{1+z}\,(-Q-zI)^{-1}\,\psi\;dz \in D\,(P)\,. \end{cases} \tag{21}$$

Now, we will show that $D(L) \subset D(M)$. Let $\psi \in D(L)$, then we have

$$\begin{aligned} \int_\gamma \frac{\sqrt{z}}{1+z}\,(-Q-zI)^{-1}\,\psi\;dz &= \int_\gamma \frac{\sqrt{z}}{1+z}\,(-P-zI)\,(-Q-zI)^{-1}\,(-P-zI)^{-1}\,\psi\;dz \\ &= \int_\gamma \frac{\sqrt{z}}{1+z}\,(-P+Q-Q-zI)\,(-Q-zI)^{-1}\,(-P-zI)^{-1}\,\psi\;dz \\ &= \int_\gamma \frac{\sqrt{z}}{1+z}\,(-P+Q)\,(-Q-zI)^{-1}\,(-P-zI)^{-1}\,\psi\;dz \\ &\quad + \int_\gamma \frac{\sqrt{z}}{1+z}\,(-P-zI)^{-1}\,\psi\;dz \\ &= -\int_\gamma \frac{\sqrt{z}}{1+z} B\,(-Q-zI)^{-1}\,(-P-zI)^{-1}\,\psi\;dz \\ &\quad + \int_\gamma \frac{\sqrt{z}}{1+z}\,(-P-zI)^{-1}\,\psi\;dz \\ &= -BQ^{-1}\zeta + \int_\gamma \frac{\sqrt{z}}{1+z}\,(-P-zI)^{-1}\,\psi\;dz, \end{aligned}$$

where

$$\zeta := \int_\gamma \frac{\sqrt{z}}{1+z} Q(-Q-zI)^{-1}(-P-zI)^{-1}\psi\;dz \in X.$$

Then, since $\psi \in D(L)$ and due to (21), we have

$$\int_\gamma \frac{\sqrt{z}}{1+z}(-P-zI)^{-1}\psi\;dz = Q^{-1}B\zeta + \int_\gamma \frac{\sqrt{z}}{1+z}(-Q-zI)^{-1}\psi\;dz \in D(Q) = D(P),$$

Thus, from (21), we deduce that $\psi \in D(\sqrt{-P}) = D(M)$.

Replacing P by Q in the previous proof, we show that $D(M) \subset D(L)$. □

Now, we give an explicit representation formula of the classical solution u of equation (5).

Proposition 3.7. Let $f \in L^p(a,b;X)$, $p \in (1,+\infty)$. Assume that (H_1), (H_2), (H_3), (H_4) and (H_5) hold. If u is a classical solution of (5), then there exists $K_1, K_2, K_3, K_4 \in X$, such that for all $x \in [a,b]$, we have

$$u(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 + e^{(x-a)L}K_3 + e^{(b-x)L}K_4 + F_{0,f}(x), \tag{22}$$

where $F_{0,f}$ is defined in Theorem 3.1 with $\Phi = 0 = (0,0,0,0)$.

Proof. If u is a classical solution of (5), due to Theorem 3.1, we can take the classical solution $F_{0,f}$ of (5)-(BC1) as a particular solution; *i.e.*

$$F_{0,f}(a) = F_{0,f}(b) = F''_{0,f}(a) = F''_{0,f}(b) = 0. \tag{23}$$

Then $u_{hom} := u - F_{0,f}$ is a classical solution of

$$u^{(4)}(x) + (P+Q)u''(x) + PQu(x) = 0, \quad x \in (a,b).$$

We set

$$v := -QB^{-1}u_{hom} - B^{-1}u''_{hom}, \quad \text{and} \quad w := PB^{-1}u_{hom} + B^{-1}u''_{hom}.$$

Moreover, from (20), we have

$$v'' + Pv = -B^{-1}\left(u_{hom}^{(4)} + (P+Q)u''_{hom} + PQu_{hom}\right) = 0,$$

and

$$w'' + Qw = B^{-1}\left(u_{hom}^{(4)} + (P+Q)u''_{hom} + PQu_{hom}\right) = 0.$$

Note that $u''_{hom} \in L^p(a,b;D(P))$ then, from (19), $PBu''_{hom} = BPu''_{hom}$ in $L^p(a,b;X)$. In the same way $QBu''_{hom} = BQu''_{hom}$. Moreover, since $u_{hom} \in L^p(a,b;D(PQ))$, we have

$$QPBu_{hom} = BQPu_{hom} = PQBu_{hom} = BPQu_{hom}.$$

Then, from [13], there exist $K_1, K_2, K_3, K_4 \in X$ such that

$$v(x) = e^{(x-a)M}K_1 + e^{(b-x)M}K_2 \quad \text{and} \quad w(x) = e^{(x-a)L}K_3 + e^{(b-x)L}K_4.$$

Finally, since $v + w = (P-Q)B^{-1}u_{hom} = u_{hom}$, we obtain (22). □

3.2.3 Regularity of the difference of analytic semigroups

Definition 3.8. A linear operator Λ on X, satisfies maximal regularity property $(\mathcal{MR})$ if and only if: there exists $q \in (1,+\infty)$ and $a,b \in \mathbb{R}$ with $a < b$ such that, for all $h \in L^q(a,b;X)$, there exists a unique $u \in W^{1,q}(a,b;X) \cap L^q(a,b;D(\Lambda))$ satisfying

$$\begin{cases} u'(x) = \Lambda u(x) + h(x), \quad \text{a.e. } x \in (a,b) \\ u(a) = 0. \end{cases}$$

Due to [8], Theorem 2.2, Theorem 2.4, Theorem 4.2, and [9], Theorem 3.2, we have

1. Λ satisfies $(\mathcal{MR})$ implies that Λ is the infinitesimal generator of an analytic semigroup.

2. $(\mathcal{MR})$ is independent of q, a and b.

3. $-\Lambda \in \mathrm{BIP}\,(X,\theta)$, $0 < \theta < \pi/2$ involves that Λ satisfies $(\mathcal{MR})$.

Consider Λ_1, Λ_2 and $\mathcal{B}$ three linear operators on X such that

$$\begin{cases} \Lambda_1 \text{ satisfies } (\mathcal{MR}) \\ 0 \in \rho(\Lambda_1) \\ \mathcal{B} \in \mathcal{L}(X) \\ \Lambda_1\mathcal{B} = \mathcal{B}\Lambda_1 \text{ on } D(\Lambda_1) \text{ (commutative case)} \\ \Lambda_2 = \Lambda_1 + \mathcal{B}. \end{cases}$$

Then, from [34], Theorem 3.4.1, p. 71 or [11], Chapter III section 1.3, p. 158, we deduce that Λ_2 is the infinitesimal generator of an analytic semigroup.

In the sequel, for a linear operator T on X, we set

$$D(T^\infty) = \bigcap_{n\in\mathbb{N}} D(T^n).$$

Theorem 3.9. Let $\psi \in X$. For $x \in (a,b) \subset \mathbb{R}$, with $a < b$, we set

$$u_\psi(x) = \left(e^{(x-a)\Lambda_2} - e^{(x-a)\Lambda_1}\right)\psi \quad \text{and} \quad v_\psi(x) = \left(e^{(b-x)\Lambda_2} - e^{(b-x)\Lambda_1}\right)\psi.$$

Then, for all $m, q \in \mathbb{N}\setminus\{0,1\}$, the following properties hold:

1. $u_\psi \in C^\infty((a,b];X) \cap C^0([a,b];X)$, moreover for $\ell \geqslant 1$ and $x \in (a,b]$

$$u_\psi^{(\ell)}(x) = \Lambda_1^\ell u_\psi(x) + T_\ell \Lambda_1^{\ell-1} e^{(x-a)\Lambda_2} \mathcal{B}\psi,$$

where $T_\ell = \sum\limits_{k=1}^{\ell} \Lambda_2^{k-1} \Lambda_1^{-(k-1)} \in \mathcal{L}(X)$.

Note that $\Lambda_2^0 = \Lambda_1^0 = I$ then in particular $T_1 = I$.

2. $u_\psi \in W^{1,p}(a,b;X) \cap L^p(a,b;D(\Lambda_1))$.

3. $u_\psi \in W^{m,p}(a,b;X) \cap L^p(a,b;D(\Lambda_1^m))$ if and only if

$$\mathcal{B}\psi \in \left(D\left(\Lambda_1^{m-1}\right), X\right)_{\frac{1}{(m-1)p},p},$$

4. If $u_\psi \in W^{m,p}(a,b;X) \cap L^p(a,b;D(\Lambda_1^m))$, then for all $\ell \in \{1,...,m-1\}$,

$$u_\psi^{(\ell)} \in W^{m-\ell,p}(a,b;X) \cap L^p\left(a,b;D(\Lambda_1^{m-\ell})\right).$$

5. If $\mathcal{B}$ satisfies $\mathcal{B}(X) \subset \left(D\left(\Lambda_1^{q-1}\right), X\right)_{\frac{1}{(q-1)p},p}$, with $q \in (1,+\infty)$, then for all $\psi \in X$:

$$u_\psi \in W^{q,p}(a,b;X) \cap L^p(a,b;D(\Lambda_1^q)).$$

6. If moreover $\mathcal{B}$ satisfies
   (a) $\mathcal{B}(X) \subset D(\Lambda_1^q)$ then $\Lambda_1^q \mathcal{B} \in \mathcal{L}(X)$
   (b) $0 \in \rho(\Lambda_1^q \mathcal{B})$,

   then, we have:
   (a) $\forall\, \psi \in X : u_\psi \in W^{q+1,p}(a,b;X) \cap L^p\left(a,b;D(\Lambda_1^{q+1})\right)$
   (b) Let $m \geqslant q+2$, with $q \in (1,+\infty)$. Then $u_\psi \in W^{m,p}(a,b;X) \cap L^p(a,b;D(\Lambda_1^m))$ if and only if

$$\psi \in \left(D\left(\Lambda_1^{m-q-1}\right), X\right)_{\frac{1}{(m-q-1)p},p}.$$

7. The six previous statements hold if we replace u_ψ by v_ψ.

Proof.

1. Note that for $x > a$, we have $e^{(x-a)\Lambda_1}\psi,\ e^{(x-a)\Lambda_2}\psi \in D(\Lambda_1^\infty) = D(\Lambda_2^\infty)$.

Moreover, $u_\psi \in C^\infty((a,b];X) \cap C^0([a,b];X)$. Then, for $\ell \geqslant 1$ and $x \in (a,b]$, it follows

$$\begin{aligned}
u_\psi^{(\ell)}(x) &= \Lambda_2^\ell e^{(x-a)\Lambda_2} - \Lambda_1^\ell e^{(x-a)\Lambda_1}\psi \\
&= \Lambda_1^\ell \left(e^{(x-a)\Lambda_2} - e^{(x-a)\Lambda_1}\right)\psi + \left(\Lambda_2^\ell - \Lambda_1^\ell\right) e^{(x-a)\Lambda_2}\psi \\
&= \Lambda_1^\ell u_\psi(x) + \left(\sum_{k=1}^{\ell} \Lambda_2^{k-1}\Lambda_1^{\ell-k}\right)(\Lambda_2 - \Lambda_1) e^{(x-a)\Lambda_2}\psi \\
&= \Lambda_1^\ell u_\psi(x) + \left(\sum_{k=1}^{\ell} \Lambda_2^{k-1}\Lambda_1^{\ell-k}\right) e^{(x-a)\Lambda_2}\mathcal{B}\psi \\
&= \Lambda_1^\ell u_\psi(x) + \left(\sum_{k=1}^{\ell} \Lambda_2^{k-1}\Lambda_1^{-(k-1)}\right) \Lambda_1^{\ell-1} e^{(x-a)\Lambda_2}\mathcal{B}\psi.
\end{aligned}$$

2. From the previous statement, u_ψ is a solution $C^1((a,b];X)\cap C^0([a,b];X)$ of the Cauchy problem

$$\left\{\begin{array}{lcl} u'(x) & = & \Lambda_1 u(x) + e^{(x-a)\Lambda_2}\mathcal{B}\psi, \quad x\in(a,b] \\ u(a) & = & 0. \end{array}\right.$$

Thus, for $x\in(a,b]$, u is given by the variation of constant formula

$$u_\psi(x) = \int_a^x e^{(x-s)\Lambda_1}e^{(s-a)\Lambda_2}\mathcal{B}\psi\, ds = \Lambda_1^{-1}\left[\Lambda_1\int_a^x e^{(x-s)\Lambda_1}e^{(s-a)\Lambda_2}\mathcal{B}\psi\, ds\right]. \qquad (24)$$

Moreover, we have

$$s\mapsto e^{(s-a)\Lambda_2}\mathcal{B}\psi \in C^0([a,b];X)\subset L^p(a,b;X),$$

and from [9], Theorem 3.2, p. 196, it follows

$$g_{1,\psi}: x\mapsto \Lambda_1\int_a^x e^{(x-s)\Lambda_1}e^{(s-a)\Lambda_2}\mathcal{B}\psi\, ds\in L^p(a,b;X).$$

We deduce that $u_\psi = \Lambda_1^{-1}g_{1,\psi}\in L^p(a,b;D(\Lambda_1))$ and then

$$u' = \Lambda_1 u + e^{(\cdot-a)\Lambda_2}\mathcal{B}\psi\in L^p(a,b;X).$$

3. Assume that $\mathcal{B}\psi\in\left(D\left(\Lambda_1^{m-1}\right),X\right)_{\frac{1}{(m-1)p},p}$. From (24), we have

$$u_\psi(x) = \Lambda_1^{-m}\left[\Lambda_1\int_a^x e^{(x-s)\Lambda_1}\Lambda_1^{m-1}e^{(s-a)\Lambda_2}\mathcal{B}\psi\, ds\right],\quad x\in(a,b].$$

Since $\mathcal{B}\psi\in\left(D\left(\Lambda_1^{m-1}\right),X\right)_{\frac{1}{(m-1)p},p}$, we obtain

$$s\mapsto\Lambda_1^{m-1}e^{(s-a)\Lambda_1}\mathcal{B}\psi\in L^p(a,b;X).$$

From [37], Theorem, p. 96, we deduce that

$$s\mapsto\Lambda_1^{m-1}e^{(s-a)\Lambda_2}\mathcal{B}\psi = e^{(s-a)\mathcal{B}}\Lambda_1^{m-1}e^{(s-a)\Lambda_1}\mathcal{B}\psi\in L^p(a,b;X),$$

and from [9], Theorem 3.2, p. 196, we have

$$g_{m,\psi}: x\mapsto\Lambda_1\int_a^x e^{(x-s)\Lambda_1}\Lambda_1^{m-1}e^{(s-a)\Lambda_2}\mathcal{B}\psi\, ds\in L^p(a,b;X).$$

It follows that

$$u_\psi = \Lambda_1^{-m}g_{m,\psi}\in L^p(a,b;D(\Lambda_1^m)).$$

Moreover, $u_\psi\in W^{m,p}(a,b;X)$ since $u_\psi\in C^1((a,b];X)\cap C^0([a,b];X)$ and from statement 1., for $\ell\in\{1,...,m\}$ and $x\in(a,b]$, we obtain

$$u_\psi^{(\ell)}(x) = \Lambda_1^\ell u_\psi(x) + T_\ell\Lambda_1^{\ell-1}e^{(x-a)\Lambda_2}\mathcal{B}\psi.$$

Then $u_\psi^{(\ell)}\in L^p(a,b;X)$.
Conversely, if $u_\psi\in W^{m,p}(a,b;X)\cap L^p(a,b;D(\Lambda_1^m))$ then, from Lemma 2.7, we have

$$\begin{array}{rcl}\mathcal{B}\psi = u'_\psi(a)\in\left(D\left(\Lambda_1^m\right),X\right)_{\frac{1}{mp}+\frac{1}{m},p} & = & \left(X,D\left(\Lambda_1^m\right)\right)_{1-\frac{1}{mp}-\frac{1}{m},p} \\ & = & \left(X,D\left(\Lambda_1\right)\right)_{m-\frac{1}{p}-1,p} \\ & = & \left(D\left(\Lambda_1^{m-1}\right),X\right)_{\frac{1}{(m-1)p},p}.\end{array}$$

4. Note first that, from statement 3., we have $\mathcal{B}\psi \in \left(D\left(\Lambda_1^{m-1}\right), X\right)_{\frac{1}{(m-1)p},p}$, then

$$\Lambda_1^{m-1} e^{(\cdot - a)\Lambda_1} \Lambda_2 \psi \in L^p(a, b; X).$$

Let $\ell \in \{1, ..., m-1\}$. For $x \in (a, b]$, we have

$$u_\psi^{(\ell)}(x) = \Lambda_1^\ell u_\psi(x) + T_\ell \Lambda_1^{\ell-1} e^{(x-a)\Lambda_2} \mathcal{B}\psi.$$

It follows that:

   (a) $u_\psi^{(\ell)} \in L^p(a, b; D(\Lambda_1^{m-\ell}))$ since

$$\Lambda_1^{m-\ell} u_\psi^{(\ell)} = \Lambda_1^m u_\psi(x) + T_\ell \Lambda_1^{m-1} e^{(x-a)\Lambda_2} \mathcal{B}\psi \in L^p(a, b; X).$$

   (b) $u_\psi^{(\ell)} \in W^{m-\ell,p}(a, b; X)$ since, for $k \in \{1, ..., m-\ell\}$, we have $\left(u_\psi^{(\ell)}\right)^{(k)} \in L^p(a, b; X)$ and

$$\begin{array}{rcl} \left(u_\psi^{(\ell)}\right)^{(k)} & = & u_\psi^{(\ell+k)} \\ & = & \Lambda_1^{\ell+k} u_\psi + T_{\ell+k} \Lambda_1^{\ell+k-1} e^{(\cdot-a)\Lambda_2} \mathcal{B}\psi, \\ & = & \Lambda_1^{\ell+k} u_\psi + T_{\ell+k} \Lambda_1^{\ell+k-m} \Lambda_1^{m-1} e^{(\cdot-a)\Lambda_2} \mathcal{B}\psi \in L^p(a, b; X). \end{array}$$

   Since $\Lambda_1^{\ell+k} u_\psi \in L^p(a, b; X)$ and $\ell + k \leqslant m$, we obtain $T_{\ell+k} \Lambda_1^{\ell+k-m} \in \mathcal{L}(X)$ and

$$T_{\ell+k} \Lambda_1^{\ell+k-m} \Lambda_1^{m-1} e^{(\cdot-a)\Lambda_2} \mathcal{B}\psi \in L^p(a, b; X).$$

5. If $\psi \in X$, then $\mathcal{B}\psi \in \left(D\left(\Lambda_1^{q-1}\right), X\right)_{\frac{1}{(q-1)p},p}$ and from statement 3., we obtain the result.

6. For statement a) it suffice to note that if $\psi \in X$, then

$$\mathcal{B}\psi \in D(\Lambda_1^q) \subset D\left(\Lambda_1^{q-1}\right) \quad \text{and} \quad \Lambda_1^{q-1} \mathcal{B}\psi = D(\Lambda_1) \subset (D(\Lambda_1), X)_{\frac{1}{p},p}.$$

From statement 3., $u_\psi \in W^{q+1,p}(a, b; X) \cap L^p\left(a, b; D(\Lambda_1^{q+1})\right)$.

For statement b), if $\psi \in \left(D\left(\Lambda_1^{m-q-1}\right), X\right)_{\frac{1}{(m-q-1)p},p}$ then, from the reiteration property described in Remark 2.6, we have

$$\psi \in D\left(\Lambda_1^{m-q-2}\right) \quad \text{and} \quad \Lambda_1^{m-q-2} \psi \in (D(\Lambda_1), X)_{\frac{1}{p},p}.$$

It follows that

$$\mathcal{B}\psi = \Lambda_1^q \mathcal{B} \Lambda_1^{-q} \psi \in D\left(\Lambda_1^{m-2}\right),$$

and

$$\Lambda_1^{m-2} \mathcal{B}\psi = \Lambda_1^{m-2} \Lambda_1^q \mathcal{B} \Lambda_1^{-q} \psi = \Lambda_1^q \mathcal{B} \Lambda_1^{m-q-2} \psi \in (D(\Lambda_1), X)_{\frac{1}{p},p}.$$

Then, from statement 3., we obtain that $u_\psi \in W^{m,p}(a, b; X) \cap L^p(a, b; D(\Lambda_1^m))$.

Conversely, if $u_\psi \in W^{m,p}(a, b; X) \cap L^p(a, b; D(\Lambda_1^m))$, then, from statement 3., we have $\mathcal{B}\psi \in \left(D\left(\Lambda_1^{m-1}\right), X\right)_{\frac{1}{(m-1)p},p}$. Hence, from Remark 2.6, we deduce that

$$\mathcal{B}\psi \in D\left(\Lambda_1^{m-2}\right) \quad \text{and} \quad \Lambda_1^{m-2} \mathcal{B}\psi \in (D(\Lambda_1), X)_{\frac{1}{p},p}.$$

Then, there exists $\chi \in (D(\Lambda_1), X)_{\frac{1}{p},p}$ such that $\mathcal{B}\psi = \Lambda_1^{-m+2}\chi$, it follows

$$\psi = (\Lambda_1^q \mathcal{B})^{-1} \Lambda_1^q \mathcal{B}\psi = (\Lambda_1^q \mathcal{B})^{-1} \Lambda_1^{-m+q+2}\chi \in D\left(\Lambda_1^{m-q-2}\right),$$

and

$$\Lambda_1^{m-q-2}\psi = (\Lambda_1^q \mathcal{B})^{-1} \chi \in (D(\Lambda_1), X)_{\frac{1}{p},p},$$

which gives $\psi \in \left(D\left(\Lambda_1^{m-q-1}\right), X\right)_{\frac{1}{(m-q-1)p},p}$.

7. It suffice to write $v_\psi(x) = u_\psi(b + a - x)$. Then, v_ψ satisfies the properties of u_ψ.

$\square$

Corollary 3.10. Assume that (H_1), (H_2), (H_3), (H_4) and (H_5) are satisfied. Let $\psi \in X$. For $x \in (a, b) \subset \mathbb{R}$, we set

$$u_\psi(x) = \left(e^{(x-a)L} - e^{(x-a)M}\right)\psi \quad \text{and} \quad v_\psi(x) = \left(e^{(b-x)L} - e^{(b-x)M}\right)\psi.$$

Let $m \geqslant 3$, then

1. $\forall\ \psi \in X : u_\psi \in W^{2,p}(a, b; X) \cap L^p(a, b; D(M^2))$.
2. $u_\psi \in W^{m,p}(a, b; X) \cap L^p(a, b; D(M^m)) \iff \psi \in (D(M^{m-2}), X)_{\frac{1}{(m-2)p},p}$.

   In this case, for all $\ell \in \{1, ..., m-1\}$

   $$u_\psi^{(\ell)} \in W^{m-\ell,p}(a, b; X) \cap L^p\left(a, b; D(M^{m-\ell})\right).$$

   In particular,

   $$u_\psi \in W^{4,p}(a, b; X) \cap L^p\left(a, b; D(M^4)\right) \iff \psi \in (D(M), X)_{1+\frac{1}{p},p},$$

   and in this case $u''_\psi \in L^p(a, b; D(M^2))$.
3. The previous statement holds true if we replace u_ψ by v_ψ.

Proof. We set $\mathcal{B} = B(L + M)^{-1}$. Moreover, $\mathcal{B}$ satisfies

$$\begin{cases} 1)\ \mathcal{B}(X) \subset D(L + M) = D(M) \text{ then } M\mathcal{B} \in \mathcal{L}(X). \\ 2)\ 0 \in \rho(M\mathcal{B}) \text{ with } (M\mathcal{B})^{-1} = (L + M) M^{-1}B^{-1}. \end{cases}$$

From (17), we have $L = M + B(L + M)^{-1}$. Then, setting $\Lambda_2 = L$, $\Lambda_1 = M$ and $\mathcal{B} = B(L + M)^{-1}$ in statement 6. of Theorem 3.9, we obtain statements 1. and 2. Moreover, from statement 7. of Theorem 3.9, we obtain statement 3. $\square$

3.3 Proof of Theorem 3.2

We first give a useful remark concerning the regularity.

Remark 3.11. From Lemma 2.7, if $u \in W^{4,p}(a, b; X) \cap L^p(a, b; D(M^4))$ then, for $s \in [a, b]$, we obtain

$$u(s) \in (D(M^4), X)_{\frac{1}{4p},p}, \quad u'(s) \in (D(M^4), X)_{\frac{1}{4}+\frac{1}{4p},p} \quad \text{and} \quad u''(s) \in (D(M^4), X)_{\frac{1}{2}+\frac{1}{4p},p}.$$

Moreover, from Remark 2.6, for $s \in [a, b]$, we deduce that

$$u(s) \in (D(M), X)_{3+\frac{1}{p},p}, \quad u'(s) \in (D(M), X)_{2+\frac{1}{p},p} \quad \text{and} \quad u''(s) \in (D(M), X)_{1+\frac{1}{p},p}.$$

In the same way, we obtain the following equalities:

$$\begin{cases} (D(M), X)_{3+\frac{1}{p},p} = (D(P), X)_{1+\frac{1}{2p},p}, \\ (D(M), X)_{2+\frac{1}{p},p} = (D(P), X)_{1+\frac{1}{2}+\frac{1}{2p},p}, \\ (D(M), X)_{1+\frac{1}{p},p} = (D(P), X)_{\frac{1}{2p},p}. \end{cases} \tag{25}$$

Thus, we only have to prove the converse implications in Theorem 3.2.

3.3.1 Proof of 1. of Theorem 3.2 (Boundary Conditions (BC2))

Assume that (H_1), (H_2), (H_3), (H_4), (H_5) and (12) hold.

If u is a classical solution of (5)-(BC2), then from Proposition 3.7, u satisfies (22). We set

$$\alpha_1 = \frac{K_1 - K_2}{2}, \quad \alpha_2 = \frac{K_3 - K_4}{2}, \quad \alpha_3 = \frac{K_1 + K_2}{2} \quad \text{and} \quad \alpha_4 = \frac{K_3 + K_4}{2}. \tag{26}$$

Then, for $a.e.\ x \in (a,b)$, u is given by

$$\begin{aligned} u(x) \quad = \quad & \left(e^{(x-a)M} - e^{(b-x)M}\right)\alpha_1 + \left(e^{(x-a)L} - e^{(b-x)L}\right)\alpha_2 \\ & + \left(e^{(x-a)M} + e^{(b-x)M}\right)\alpha_3 + \left(e^{(x-a)L} + e^{(b-x)L}\right)\alpha_4 + F_{0,f}(x). \end{aligned} \tag{27}$$

Following the same steps as those used in the proof of Theorem 2.5, p. 365 in [21] (where we replace kI by B), we obtain

$$\left\{ \begin{array}{lcl} \alpha_1 & = & \left(I + e^{cM}\right)^{-1} \left(M^{-1}\tilde{\varphi}_1 - \left(I + e^{cL}\right) LM^{-1}\alpha_2\right) \\ \alpha_2 & = & \left(I - e^{cL}\right)^{-1} B^{-1} \left(\dfrac{\varphi_3 - \varphi_4}{2}\right) \\ \alpha_3 & = & \left(I - e^{cM}\right)^{-1} \left(M^{-1}\tilde{\varphi}_2 - \left(I - e^{cL}\right) LM^{-1}\alpha_4\right) \\ \alpha_4 & = & \left(I + e^{cL}\right)^{-1} B^{-1} \left(\dfrac{\varphi_3 + \varphi_4}{2}\right), \end{array} \right. \tag{28}$$

where

$$\tilde{\varphi}_1 \ := \ \frac{\varphi_1 + \varphi_2 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 \ := \ \frac{\varphi_1 - \varphi_2 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \tag{29}$$

Now, thanks to Lemma 2.8, Lemma 2.9, Corollary 3.10 and using again the same method as in [21], we obtain that u is the unique classical solution of (5)-(BC2).

3.3.2 Proof of 2. of Theorem 3.2 (Boundary Conditions (BC3))

Assume that (H_1), (H_2), (H_3), (H_4), (H_5), (H_6) and (13) hold.

As previously, following the same steps as those used in the proof of Theorem 2.5, p. 365 in [21] (where we replace kI by B) and using Lemma 2.8, Lemma 2.9 and Corollary 3.10, we deduce that u is the unique classical solution of (5)-(BC3), given by (22) where

$$\left\{ \begin{array}{lcl} \alpha_1 & = & \dfrac{1}{2}B^{-1}(L+M)U^{-1}\left[L(I+e^{cL})(\varphi_1 - \varphi_2) - 2(I - e^{cL})\tilde{\varphi}_1\right] \\ \alpha_2 & = & -\dfrac{1}{2}B^{-1}(L+M)U^{-1}\left[M(I+e^{cM})(\varphi_1 - \varphi_2) - 2(I - e^{cM})\tilde{\varphi}_1\right] \\ \alpha_3 & = & \dfrac{1}{2}B^{-1}(L+M)V^{-1}\left[L(I-e^{cL})(\varphi_1 + \varphi_2) - 2(I + e^{cL})\tilde{\varphi}_2\right] \\ \alpha_4 & = & -\dfrac{1}{2}B^{-1}(L+M)V^{-1}\left[M(I-e^{cM})(\varphi_1 + \varphi_2) - 2(I + e^{cM})\tilde{\varphi}_2\right], \end{array} \right. \tag{30}$$

with

$$\tilde{\varphi}_1 \ := \ \frac{\varphi_3 + \varphi_4 - F'_{0,f}(a) - F'_{0,f}(b)}{2} \quad \text{and} \quad \tilde{\varphi}_2 \ := \ \frac{\varphi_3 - \varphi_4 - F'_{0,f}(a) + F'_{0,f}(b)}{2}. \tag{31}$$

3.3.3 Proof of 3. of Theorem 3.2 (Boundary Conditions (BC4))

Assume that (H_1), (H_2), (H_3), (H_4), (H_5), (H_6) and (14) hold. We proceed as in the previous proof.

Following the same steps as those used in the proof of Theorem 2.5, p. 365 in [21] (where we replace kI by B) and using Lemma 2.8, Lemma 2.9 and Corollary 3.10, we deduce that u is the unique classical solution of (5)-(BC4), given by (22) with

$$\left\{\begin{array}{lcl} \alpha_1 & = & \frac{1}{2}B^{-1}(L+M)V^{-1}\left[2(I-e^{cL})LM^{-1}\tilde{\varphi}_1-(I+e^{cL})M^{-1}(\varphi_3-\varphi_4)\right] \\ \alpha_2 & = & -\frac{1}{2}B^{-1}(L+M)V^{-1}\left[2(I-e^{cM})ML^{-1}\tilde{\varphi}_1-(I+e^{cM})L^{-1}(\varphi_3-\varphi_4)\right] \\ \alpha_3 & = & \frac{1}{2}B^{-1}(L+M)U^{-1}\left[2(I+e^{cL})LM^{-1}\tilde{\varphi}_2-(I-e^{cL})M^{-1}(\varphi_3+\varphi_4)\right] \\ \alpha_4 & = & -\frac{1}{2}B^{-1}(L+M)U^{-1}\left[2(I+e^{cM})ML^{-1}\tilde{\varphi}_2-(I-e^{cM})L^{-1}(\varphi_3+\varphi_4)\right], \end{array}\right. \tag{32}$$

where $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are given by (29).

4 Back to the parabolic problem

Let X be a complex Banach space and $\mathcal{A}_i$, $i=1,2,3,4,5$, the linear operator defined by

$$\left\{\begin{array}{lcl} D(\mathcal{A}_i) & = & \{u\in W^{4,p}(a,b;X)\cap L^p(a,b;D(A^2)) \text{ and } u''\in L^p(a,b;D(A)) : (\mathrm{BCi})_0\} \\ \left[\mathcal{A}_i u\right](x) & = & -u^{(4)}(x)-(2A-kI)u''(x)-(A^2-kA)u(x), \quad u\in D(\mathcal{A}_i),\ x\in(a,b), \end{array}\right. \tag{33}$$

where $k\in\mathbb{R}$ and $(\mathrm{BCi})_0$ represents the boundary conditions (BCi), $i=1,2,3,4,5$, with $\varphi_j=0$, $j=1,2,3,4$ and wherein A_0 is replaced by a more general closed linear operator A satisfying the assumptions below.

Then, we will study the spectral properties of $\mathcal{A}_i$ in order to solve the Cauchy problem (2), where f and v_0 are in appropriate spaces.

4.1 Assumptions and main results

4.1.1 Assumptions

Let A be a closed linear operator and assume

$$\begin{array}{ll} (\mathcal{H}_1) & X \text{ is a UMD space,} \\ (\mathcal{H}_2) & 0\in\rho(A), \\ (\mathcal{H}_3) & -A\in \mathrm{BIP}(X,\theta_A), \quad \text{for } \theta_A\in[0,\pi/2), \\ (\mathcal{H}_4) & [k,+\infty)\in\rho(A). \end{array}$$

In some results, for the boundary conditions (BC3) and (BC4), we may need a supplementary hypothesis:

$$(\mathcal{H}_5) \quad -A\in \mathrm{Sect}(0).$$

4.1.2 Main results

Proposition 4.1. Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then

1. for $i=1,2,5$, we have

$$-\mathcal{A}_i+\frac{k^2}{4}I\in \mathrm{Sect}\,(2\theta_A),$$

2. for $i = 3, 4$, we have

$$\begin{cases} -\mathcal{A}_i + \dfrac{k^2}{4} I + rI \in \mathrm{Sect}\left(\dfrac{\pi}{2}\right), & \text{if } 2\theta_A \in \left[0, \dfrac{\pi}{2}\right) \\ -\mathcal{A}_i + \dfrac{k^2}{4} I + rI \in \mathrm{Sect}\left(2\theta_A\right), & \text{if } 2\theta_A \in \left[\dfrac{\pi}{2}, \pi\right), \end{cases}$$

where $r > 0$ is defined in Proposition 4.10.

Moreover, there exist $r' > r$ and $\theta_0 > 0$, such that

$$\begin{cases} -\mathcal{A}_i + \dfrac{k^2}{4} I + r'I \in \mathrm{Sect}\left(2\theta_A\right), & \text{if } 2\theta_A \in \left(0, \dfrac{\pi}{2}\right) \\ -\mathcal{A}_i + \dfrac{k^2}{4} I + r'I \in \mathrm{Sect}\left(\theta_0\right), & \text{if } 2\theta_A = 0. \end{cases}$$

Theorem 4.2. Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then, for $i = 1, 2, 3, 4, 5$, if $\theta_A < \pi/4$, operator $\mathcal{A}_i$ is the infinitesimal generator of an analytic strongly continuous semigroup $\left(e^{t\mathcal{A}_i}\right)_{t \geqslant 0}$.

Remark 4.3. For $i = 1, 2, 5$ and $\theta_A < \pi/4$, from Proposition 4.1, we have

$$\exists\, M_i \geqslant 1 : \forall\, t \geqslant 0, \quad \left\| e^{t\mathcal{A}_i} \right\|_{\mathcal{L}(X)} \leqslant M_i \, e^{t \frac{k^2}{4}}.$$

First of all, we have to study the spectral properties of $\mathcal{A}_i$. Thus, we focus on the resolvent set of $-\mathcal{A}_i$. To this end, we analyse the equation $(-\mathcal{A}_i - \lambda I)u = f$.

4.2 Study of the resolvent set

Let $\lambda \in \mathbb{C}$ and fix $i \in \{1, 2, 3, 4, 5\}$. By definition $\lambda \in \rho(\mathcal{A}_i)$ means that the following equation

$$u^{(4)}(x) + (2A - kI)u''(x) + (A^2 - kA - \lambda I)u(x) = f(x), \quad x \in (a, b), \tag{34}$$

supplemented by the boundary conditions $(\mathrm{BCi})_0$ admits a unique solution u in $D(\mathcal{A}_i)$. Since A satisfies the first four previous assumptions, we define

$$A_{k/2} = A - \frac{k}{2} I.$$

Let $\lambda \in \mathbb{C} \setminus (-k^2/4, +\infty)$ which means that $-\lambda - \dfrac{k^2}{4} \in \mathbb{C} \setminus (-\infty, 0)$. We set

$$P_\lambda = A_{k/2} + i\sqrt{-\lambda - \frac{k^2}{4}} I \quad \text{and} \quad Q_\lambda = A_{k/2} - i\sqrt{-\lambda - \frac{k^2}{4}} I, \tag{35}$$

so that we can rewrite equation (34) as

$$u^{(4)}(x) + (P_\lambda + Q_\lambda)u''(x) + P_\lambda Q_\lambda u(x) = f(x), \quad x \in (a, b),$$

and use results of section 3.

We state three technical lemmas which allow us to justify the other results of this section. We first recall that if $z \in \mathbb{C}$, then $\arg(z)$ is the unique argument of z in $(-\pi, \pi]$.

Lemma 4.4. Let $\lambda \in \mathbb{C}$. The two following statements are equivalent

- $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$,

- $\lambda \neq -\dfrac{k^2}{4}$ and $\left|\arg\left(-\lambda - \dfrac{k^2}{4}\right) \pm \pi\right| < 2\left(\pi - \theta_A\right).$

Proof. Let $\lambda \in \mathbb{C}$, then

$$\left|\arg\left(-\lambda - \frac{k^2}{4}\right) \pm \pi\right| < 2\left(\pi - \theta_A\right),$$

is equivalent to

$$\arg\left(-\lambda - \frac{k^2}{4}\right) < \pi - 2\theta_A \quad \text{and} \quad -\arg\left(-\lambda - \frac{k^2}{4}\right) < \pi - 2\theta_A. \tag{36}$$

Now, it remains to study the two following cases.

- First case : $\arg\left(-\lambda - \dfrac{k^2}{4}\right) \geqslant 0.$

  Here, (36) is equivalent to

$$0 \leqslant \arg\left(-\lambda - \frac{k^2}{4}\right) < \pi - 2\theta_A,$$

  and using $\arg\left(\lambda + \dfrac{k^2}{4}\right) = \arg\left(-\lambda - \dfrac{k^2}{4}\right) - \pi$, then (36) becomes

$$-\pi \leqslant \arg\left(\lambda + \frac{k^2}{4}\right) < -2\theta_A.$$

- Second case : $\arg\left(-\lambda - \dfrac{k^2}{4}\right) < 0.$

  Now, (36) writes

$$0 \leqslant -\arg\left(-\lambda - \frac{k^2}{4}\right) < \pi - 2\theta_A,$$

  and using $\arg\left(\lambda + \dfrac{k^2}{4}\right) = \arg\left(-\lambda - \dfrac{k^2}{4}\right) + \pi$, then (36) becomes

$$2\theta_A < \arg\left(\lambda + \frac{k^2}{4}\right) \leqslant \pi.$$

Finally, (36) is equivalent to $\lambda + \dfrac{k^2}{4} \in \mathbb{C} \setminus \overline{S_{2\theta_A}}$, which gives the result. ☐

Lemma 4.5. Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Let $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$. Then, we have

1. $\left|\arg\left(\pm i\sqrt{-\lambda - \dfrac{k^2}{4}}\right)\right| = \dfrac{\left|\arg\left(-\lambda - \frac{k^2}{4}\right) \pm \pi\right|}{2} < \pi - \theta_A.$

2. $-P_\lambda \in \mathrm{BIP}\left(X, \theta_1\right)$ and $-Q_\lambda \in \mathrm{BIP}\left(X, \theta_2\right)$, with $\theta_1, \theta_2 \in [0, \pi - \theta_A)$, where

$$\theta_1 := \max\left(\theta_A, \frac{\left|\arg\left(-\lambda - \frac{k^2}{4}\right) + \pi\right|}{2}\right) \quad \text{and} \quad \theta_2 := \max\left(\theta_A, \frac{\left|\arg\left(-\lambda - \frac{k^2}{4}\right) - \pi\right|}{2}\right).$$

Proof.

1. Since $\theta_A < \pi/2$, the result follows from Lemma 4.4.

2. From $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and [1], Theorem 2.3, p. 69, we have

$$-A_{k/2} \in \mathrm{BIP}(X, \theta_A),$$

hence, if $\lambda = -k^2/4$, we obtain

$$-P_{-k^2/4} = -Q_{-k^2/4} = -A_{k/2} \in \mathrm{BIP}\,(X, \theta_A).$$

Moreover, if $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$, then

$$\left|\arg\left(\pm i\sqrt{-\lambda - \frac{k^2}{4}}\right)\right| = \left|\pm\frac{\pi}{2} + \frac{\arg\left(-\lambda - \frac{k^2}{4}\right)}{2}\right| \in (0, \pi), \tag{37}$$

so $\pm i\sqrt{-\lambda - \frac{k^2}{4}} \notin \mathbb{R}$.

Finally, $-A_{k/2} \in \mathrm{BIP}\,(X, \theta_A)$ and $\pm i\sqrt{-\lambda - \frac{k^2}{4}} \in \mathbb{C} \setminus (-\infty, 0)$. Moreover, from (37) and Lemma 4.4, we have

$$\theta_A + \left|\arg\left(\pm i\sqrt{-\lambda - \frac{k^2}{4}}\right)\right| < \pi, \tag{38}$$

thus, from [27], Theorem 2.4, p. 408, we obtain

$$-P_\lambda \in \mathrm{BIP}(X, \theta_1) \quad \text{and} \quad -Q_\lambda \in \mathrm{BIP}(X, \theta_2).$$

□

Now, in order to use [10] in the next proof, we first need to give the following remark.

Remark 4.6. In the sequel, we will use results of [10] in which the authors use operators of type φ instead of sectorial operators. For $\varphi \in (0, \pi)$. A closed linear operator $T : D(T) \subset X \longrightarrow X$ is said of type φ with bound C if and only if $\overline{S_\varphi} \subset \rho(-T)$ and

$$\forall \lambda \in \overline{S_\varphi}, \qquad \left\|(T + \lambda I)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{1 + |\lambda|}.$$

It is clear that if T is of type φ, then $T \in \mathrm{Sect}(\pi - \varphi)$. More precisely, the two notions are linked by the equivalence of the two following assertions:

1. $T \in \mathrm{Sect}(\theta_T)$ with $\theta_T \in [0, \pi)$ and $0 \in \rho(T)$,

2. $\forall\, \varepsilon \in (0, \pi - \theta_T)$, T is of type $\varphi = \pi - \theta_T - \varepsilon \in (0, \pi)$.

Lemma 4.7. Let $k \in \mathbb{R}$ and $\lambda \in -k^2/4 + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$ and assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then $0 \in \rho(P_\lambda) \cap \rho(Q_\lambda)$ and there exists $C > 0$, independent of λ, such that

$$\left\|M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)} \leqslant C \quad \text{and} \quad \left\|L_\lambda M_\lambda^{-1}\right\|_{\mathcal{L}(X)} \leqslant C,$$

where $M_\lambda := -\sqrt{-P_\lambda}$ and $L_\lambda := -\sqrt{-Q_\lambda}$.

Proof. Since $-A_{k/2} \in \mathrm{BIP}\,(X,\theta_A)$, its spectral angle $\psi_{A,k}$ satisfies $0 \leqslant \psi_{A,k} \leqslant \theta_A$ (see [30], p. 218). Moreover, $\psi_{\lambda,k}$, the spectral angle of $i\sqrt{-\lambda-\frac{k^2}{4}}\,I$, satisfies $\psi_{\lambda,k} = \left|\arg\left(i\sqrt{-\lambda-\frac{k^2}{4}}\right)\right|$. Furthermore, from (38), it follows that

$$\psi_{A,k} + \psi_{\lambda,k} \leqslant \theta_A + \left|\arg\left(i\sqrt{-\lambda-\frac{k^2}{4}}\right)\right| < \pi.$$

Then due to Theorem 8.3 (iv), p. 218 in [30], we have $0 \in \rho(P_\lambda)$. In the same way, we obtain $0 \in \rho(Q_\lambda)$. Thus, from Lemma 4.5, we deduce $\sqrt{-P_\lambda}$ and $\sqrt{-Q_\lambda}$ are well defined and invertible with bounded inverse. Since we have

$$M_\lambda L_\lambda^{-1} = M_\lambda^2 M_\lambda^{-1} L_\lambda^{-1} = -P_\lambda M_\lambda^{-1} L_\lambda^{-1},$$

we deduce that

$$\begin{array}{rcl}
M_\lambda L_\lambda^{-1} & = & \left(-A_{k/2} - i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right) M_\lambda^{-1} L_\lambda^{-1} \\[2ex]
& = & -A_{k/2} M_\lambda^{-1} L_\lambda^{-1} - i\sqrt{-\lambda-\dfrac{k^2}{4}}\,M_\lambda^{-1} L_\lambda^{-1} \\[2ex]
& = & \sqrt{-A_{k/2}}\,M_\lambda^{-1}\sqrt{-A_{k/2}}\,L_\lambda^{-1} - i\sqrt{-\lambda-\dfrac{k^2}{4}}\,M_\lambda^{-1} L_\lambda^{-1} \\[2ex]
& = & \sqrt{-A_{k/2}}\left(-A_{k/2} - i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}}\sqrt{-A_{k/2}}\left(-A_{k/2} + i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}} \\[2ex]
& & -i\sqrt{-\lambda-\dfrac{k^2}{4}}\left(-A_{k/2} - i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}}\left(-A_{k/2} + i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}}.
\end{array}$$

Thus, it follows that

$$\begin{array}{rcl}
\left\|M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)} & \leqslant & \left\|\sqrt{-A_{k/2}}\left(-A_{k/2} - i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}}\right\|_{\mathcal{L}(X)} \\[2ex]
& & \times\left\|\sqrt{-A_{k/2}}\left(-A_{k/2} + i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}}\right\|_{\mathcal{L}(X)} \\[2ex]
& & +\sqrt{\left|\lambda+\dfrac{k^2}{4}\right|}\left\|\left(-A_{k/2} - i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}}\right\|_{\mathcal{L}(X)} \\[2ex]
& & \times\left\|\left(-A_{k/2} + i\sqrt{-\lambda-\dfrac{k^2}{4}}\,I\right)^{-\frac{1}{2}}\right\|_{\mathcal{L}(X)}.
\end{array}$$

Thus, due to $(\mathcal{H}_2)$ and Remark 4.6, we can apply [10], Lemma 2.6 statement a), p. 104: there exist $C_1, C_2 > 0$, which are independent of λ, such that

$$\left\|\sqrt{-A_{k/2}}\left(-A_{k/2} \pm i\sqrt{-\lambda-\frac{k^2}{4}}\,I\right)^{-\frac{1}{2}}\right\| \leqslant C_1, \tag{39}$$

and

$$\left\| \left(-A_{k/2} \pm i\sqrt{-\lambda - \frac{k^2}{4}}\, I \right)^{-\frac{1}{2}} \right\|_{\mathcal{L}(X)} \leqslant \frac{C_2}{\sqrt{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}}}. \tag{40}$$

Therefore, (39) and (40) give us the expected estimate for $\|M_\lambda L_\lambda^{-1}\|$. In the same way, replacing M_λ by L_λ and L_λ by M_λ, we obtain the symmetric result for $\|L_\lambda M_\lambda^{-1}\|$. □

Now, we adapt a useful technical lemma from [10].

Lemma 4.8 ([10])**.** Let $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$ and assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then, the two analytic semigroups $\left(e^{-t\sqrt{-P_\lambda}}\right)_{t\geqslant 0}$ and $\left(e^{-t\sqrt{-Q_\lambda}}\right)_{t\geqslant 0}$, are well defined.

Moreover, let $\alpha \in \mathbb{R}$ and fix $t_0 > 0$, there exist $K > 0$ and $\tilde{\omega} > 0$, such that for any $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$, we have

$$\left\|(-P_\lambda)^\alpha e^{-t_0\sqrt{-P_\lambda}}\right\|_{\mathcal{L}(X)} \leqslant K e^{-t_0\tilde{\omega}\left|\lambda + \frac{k^2}{4}\right|^{1/4}} \quad \text{and} \quad \left\|(-Q_\lambda)^\alpha e^{-t_0\sqrt{-Q_\lambda}}\right\|_{\mathcal{L}(X)} \leqslant K e^{-t_0\tilde{\omega}\left|\lambda + \frac{k^2}{4}\right|^{1/4}}.$$

Proof. Let $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$. From Lemma 4.5 and [18], Proposition 3.1.2, p. 63, we deduce that

$$\sqrt{-P_\lambda} \in \text{BIP}\,(X, \theta_1/2) \quad \text{and} \quad \sqrt{-Q_\lambda} \in \text{BIP}\,(X, \theta_2/2),$$

with $\theta_1/2, \theta_2/2 \in [0, \pi/2)$. Then, we deduce that $-\sqrt{-P_\lambda}$ and $-\sqrt{-Q_\lambda}$ are the infinitesimal generators of the two analytic semigroups

$$\left(e^{-t\sqrt{-P_\lambda}}\right)_{t\geqslant 0} = \left(e^{-t\sqrt{-A_{k/2} - i\sqrt{-\lambda - \frac{k^2}{4}}\, I}}\right)_{t\geqslant 0},$$

and

$$\left(e^{-t\sqrt{-Q_\lambda}}\right)_{t\geqslant 0} = \left(e^{-t\sqrt{-A_{k/2} + i\sqrt{-\lambda - \frac{k^2}{4}}\, I}}\right)_{t\geqslant 0}.$$

Moreover, let $\alpha \in \mathbb{R}$ and fix $t_0 > 0$, from [10], Lemma 2.6, statement b), p. 104, we deduce that there exist $K \geqslant 1$ and $\tilde{\omega} > 0$, such that

$$\begin{aligned} \left\|(-P_\lambda)^\alpha e^{-t_0\sqrt{-P_\lambda}}\right\|_{\mathcal{L}(X)} &= \left\| \left(-A_{k/2} - i\sqrt{-\lambda - \frac{k^2}{4}} I \right)^\alpha e^{-t_0\sqrt{-A_{k/2} - i\sqrt{-\lambda - \frac{k^2}{4}} I}} \right\|_{\mathcal{L}(X)} \\ &\leqslant K e^{-t_0\tilde{\omega}\sqrt{\left|i\sqrt{-\lambda - \frac{k^2}{4}}\right|}}, \end{aligned}$$

and

$$\begin{aligned} \left\|(-Q_\lambda)^\alpha e^{-t_0\sqrt{-Q_\lambda}}\right\|_{\mathcal{L}(X)} &= \left\| \left(-A_{k/2} + i\sqrt{-\lambda - \frac{k^2}{4}} I \right)^\alpha e^{-t_0\sqrt{-A_{k/2} + i\sqrt{-\lambda - \frac{k^2}{4}} I}} \right\|_{\mathcal{L}(X)} \\ &\leqslant K e^{-t_0\tilde{\omega}\sqrt{\left|i\sqrt{-\lambda - \frac{k^2}{4}}\right|}}, \end{aligned}$$

with

$$Ke^{-t_0\tilde{\omega}\sqrt{\left|i\sqrt{-\lambda-\frac{k^2}{4}}\right|}} = Ke^{-t_0\tilde{\omega}\sqrt{\sqrt{\left|\lambda+\frac{k^2}{4}\right|}}} = Ke^{-t_0\tilde{\omega}\left|\lambda+\frac{k^2}{4}\right|^{1/4}}.$$

□

Proposition 4.9. Let $i = 1, 2, 5$ and assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ hold. Then, we have

$$\{0\} \cup \left\{-\frac{k^2}{4}\right\} \cup \left(-\frac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)\right) \subset \rho(-\mathcal{A}_i).$$

Proof.

- If $\lambda = 0$, we obtain, for $k \neq 0$, that $0 \in \rho(\mathcal{A}_i)$, from [21], Theorem 2.2, p. 355 and Theorem 2.5, p. 356-357, by taking

$$\begin{cases} P_\lambda = A - kI & \text{and} \quad Q_\lambda = A, & \text{if } k > 0, \\ P_\lambda = A & \text{and} \quad Q_\lambda = A - kI, & \text{if } k < 0, \end{cases}$$

  Moreover, if $k = 0$, from [35], Theorem 2.6 and Theorem 2.8, then $0 \in \rho(\mathcal{A}_i)$.

- If $\lambda = -\dfrac{k^2}{4}$, then from Lemma 4.5, $-P_{-k^2/4} = -Q_{-k^2/4} = -A_{k/2} \in \mathrm{BIP}\,(X, \theta)$, where $\theta = \max(\theta_1, \theta_2)$. Thus, from [35], Theorem 2.6 and Theorem 2.8, we have $-k^2/4 \in \rho(-\mathcal{A}_i)$.

- Let $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$. Then, using P_λ and Q_λ defined by (35), we obtain that

$$B_\lambda := 2i\sqrt{-\lambda - \frac{k^2}{4}}I \in \mathcal{L}(X), \tag{41}$$

  is invertible with bounded inverse. Moreover, we have $P_\lambda = Q_\lambda + B_\lambda$.

  We are now in position to apply the results of section 3 with P, Q replaced by P_λ, Q_λ. From Lemma 4.5, it follows that assumptions (H_1), (H_2), (H_3), (H_4) and (H_5) of section 3.1 are satisfied. Then, from Theorem 3.1 and Theorem 3.2, there exists a unique classical solution of (34)-$(\mathrm{BCi})_0$. Thus, we deduce that have $\lambda \in \rho(-\mathcal{A}_i)$.

□

Proposition 4.10. Let $i = 3, 4$ and assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ hold. Then, there exists $r > 0$, such that

$$\left\{-\frac{k^2}{4}\right\} \cup \left(-\frac{k^2}{4} + \left(\mathbb{C} \setminus \left(\overline{B(0,r)} \cup \overline{S_{2\theta_A}}\right)\right)\right) \subset \rho(-\mathcal{A}_i).$$

Moreover, if in addition, we assume $(\mathcal{H}_5)$, we obtain $0 \in \rho(-\mathcal{A}_i)$.

Proof. As in the proof of Proposition 4.9, assumptions (H_1), (H_2), (H_3), (H_4) and (H_5) of section 3.1 are satisfied and also $-\dfrac{k^2}{4} \in \rho(-\mathcal{A}_i)$.

Now, we adapt notations of section 3.1, replacing L, M, U, V, T^-, T^+, B and M_B by L_λ, M_λ, U_λ, V_λ, T_λ^-, T_λ^+, B_λ and M_{B_λ}. Our aim is to show that (H_6) of section 3.1 holds with U_λ and V_λ instead of U and V. To this end, we recall that $c = b - a > 0$. From Lemma 4.8, for $t_0 = c > 0$, there exist $K \geqslant 1$ and some $\tilde{\omega} > 0$, such that, for any $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$, we have

$$\begin{cases} \|M_\lambda^2 e^{cM_\lambda}\|_{\mathcal{L}(X)} &= \| - P_\lambda e^{-c\sqrt{-P_\lambda}}\|_{\mathcal{L}(X)} &\leqslant Ke^{-c\tilde{\omega}\left|\lambda+\frac{k^2}{4}\right|^{1/4}} \\ \|L_\lambda^2 e^{cL_\lambda}\|_{\mathcal{L}(X)} &= \| - Q_\lambda e^{-c\sqrt{-Q_\lambda}}\|_{\mathcal{L}(X)} &\leqslant Ke^{-c\tilde{\omega}\left|\lambda+\frac{k^2}{4}\right|^{1/4}}, \end{cases}$$

hence

$$\begin{aligned}\|e^{c(L_\lambda+M_\lambda)}\|_{\mathcal{L}(X)} = \|e^{-c(\sqrt{-Q_\lambda}+\sqrt{-P_\lambda})}\|_{\mathcal{L}(X)} &= \|e^{-c\sqrt{-Q_\lambda}}e^{-c\sqrt{-P_\lambda}}\|_{\mathcal{L}(X)} \\ &\leqslant \|e^{-c\sqrt{-Q_\lambda}}\|_{\mathcal{L}(X)}\|e^{-c\sqrt{-P_\lambda}}\|_{\mathcal{L}(X)} \\ &\leqslant K^2 e^{-2c\tilde{\omega}\left|\lambda+\frac{k^2}{4}\right|^{1/4}},\end{aligned}$$

where L_λ and M_λ are defined by (7). Moreover, from Lemma 4.7, there exists $C > 0$, which is independent of λ, such that

$$\left\{\begin{array}{lllll}\left\|(L_\lambda+M_\lambda)^2 M_\lambda^{-2}\right\|_{\mathcal{L}(X)} & \leqslant & \left\|L_\lambda M_\lambda^{-1}\right\|_{\mathcal{L}(X)}^2 + 2\left\|L_\lambda M_\lambda^{-1}\right\|_{\mathcal{L}(X)} + 1 & \leqslant & C \\ \left\|(L_\lambda+M_\lambda)^2 L_\lambda^{-2}\right\|_{\mathcal{L}(X)} & \leqslant & \left\|M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)}^2 + 2\left\|M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)} + 1 & \leqslant & C.\end{array}\right.$$

Furthermore, from (41), we have

$$\left\|B_\lambda^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{1}{2\sqrt{\left|\lambda+\frac{k^2}{4}\right|}}. \tag{42}$$

Then, due to (42), we obtain

$$\begin{aligned}M_{B_\lambda} &= \max\left(\left\|T_\lambda^-\right\|_{\mathcal{L}(X)}, \left\|T_\lambda^+\right\|_{\mathcal{L}(X)}\right) \\ &\leqslant \left\|e^{c(L_\lambda+M_\lambda)}\right\|_{\mathcal{L}(X)} + \left\|B_\lambda^{-1}(L_\lambda+M_\lambda)^2 M_\lambda^{-2}\right\|_{\mathcal{L}(X)}\left\|M_\lambda^2 e^{cM_\lambda}\right\|_{\mathcal{L}(X)} \\ &\quad + \left\|B_\lambda^{-1}(L_\lambda+M_\lambda)^2 L_\lambda^{-2}\right\|_{\mathcal{L}(X)}\left\|L_\lambda^2 e^{cL_\lambda}\right\|_{\mathcal{L}(X)} \\ &\leqslant \left\|e^{c(L_\lambda+M_\lambda)}\right\|_{\mathcal{L}(X)} + \left\|B_\lambda^{-1}\right\|_{\mathcal{L}(X)}\left\|(L_\lambda+M_\lambda)^2 M_\lambda^{-2}\right\|_{\mathcal{L}(X)}\left\|M_\lambda^2 e^{cM_\lambda}\right\|_{\mathcal{L}(X)} \\ &\quad + \left\|B_\lambda^{-1}\right\|_{\mathcal{L}(X)}\left\|(L_\lambda+M_\lambda)^2 L_\lambda^{-2}\right\|_{\mathcal{L}(X)}\left\|L_\lambda^2 e^{cL_\lambda}\right\|_{\mathcal{L}(X)} \\ &\leqslant \left\|e^{c(L_\lambda+M_\lambda)}\right\|_{\mathcal{L}(X)} + \left\|B_\lambda^{-1}\right\|_{\mathcal{L}(X)}\left(1+\left\|L_\lambda M_\lambda^{-1}\right\|_{\mathcal{L}(X)}\right)^2\left\|M_\lambda^2 e^{cM_\lambda}\right\|_{\mathcal{L}(X)} \\ &\quad + \left\|B_\lambda^{-1}\right\|_{\mathcal{L}(X)}\left(1+\left\|M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)}\right)^2\left\|L_\lambda^2 e^{cL_\lambda}\right\|_{\mathcal{L}(X)} \\ &\leqslant K^2 e^{-2c\tilde{\omega}\left|\lambda+\frac{k^2}{4}\right|^{1/4}} + \frac{CK}{\sqrt{\left|\lambda+\frac{k^2}{4}\right|}}e^{-c\tilde{\omega}\left|\lambda+\frac{k^2}{4}\right|^{1/4}}.\end{aligned}$$

Then, there exists $r > 0$ such that, for any $-\dfrac{k^2}{4} + \left(\mathbb{C}\setminus\left(\overline{B(0,r)}\cup\overline{S_{2\theta_A}}\right)\right)$, we have

$$\left|\lambda+\frac{k^2}{4}\right| \geqslant r > 0, \tag{43}$$

and

$$M_{B_\lambda} = \max\left(\left\|T_\lambda^-\right\|_{\mathcal{L}(X)}, \left\|T_\lambda^+\right\|_{\mathcal{L}(X)}\right) \leqslant \frac{1}{2} < 1.$$

For such λ, we deduce that $U_\lambda = I - T_\lambda^-$ and $V_\lambda = I - T_\lambda^+$ are invertible with bounded inverse, with

$$\|U_\lambda^{-1}\|_{\mathcal{L}(X)} \leqslant 2 \quad \text{and} \quad \|V_\lambda^{-1}\|_{\mathcal{L}(X)} \leqslant 2, \tag{44}$$

which involves that (H_6) of section 3.1 holds.

Finally, from Theorem 3.2, there exists a unique classical solution of (34)-(BCi)$_0$. Hence

$$-\frac{k^2}{4} + \left(\mathbb{C} \setminus \left(\overline{B(0,r)} \cup \overline{S_{2\theta_A}}\right)\right) \subset \rho(-\mathcal{A}_i),$$

which gives the result.

If in addition, we assume $(\mathcal{H}_5)$, then from [21], Theorem 2.2, p. 355 and Theorem 2.5, p. 356-357, we obtain that $0 \in \rho(-\mathcal{A}_i)$, $i = 3, 4$. $\square$

4.3 Norm estimates

In this section, we focus on the norm estimates of the classical solution of problem (34)-(BCi)$_0$, for $i = 1, 2, 3, 4, 5$. To this end, we adapted the following technical useful results from [14].

Lemma 4.11 ([14])**.** Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Let $f \in L^p(a, b; X)$ with $1 < p < +\infty$. Then, for all $\varphi \in (0, \pi - \theta_A)$, we have

1. $-A$ is of type φ,
2. For all $\mu \in \overline{S_\varphi} \subset \rho(-A)$ and all $x \in [a, b]$, we set

$$I_{\mu,f}(x) = \int_a^x e^{-(x-s)\sqrt{-A+\mu I}} f(s)\ ds \quad \text{and} \quad J_{\mu,f}(x) = \int_x^b e^{-(s-x)\sqrt{-A+\mu I}} f(s)\ ds.$$

   Then, for all $\mu \in \overline{S_\varphi}$, we have

$$\|I_{\mu,f}\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{1+|\mu|}} \|f\|_{L^p(a,b;X)} \quad \text{and} \quad \|J_{\mu,f}\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{1+|\mu|}} \|f\|_{L^p(a,b;X)},$$

   where C is a positive constant independent of f and μ.

Statement 1. is given by Remark 4.6 and statement 2. is proved in [14], Lemma 4.6.

Lemma 4.12. Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Let $\varphi \in (0, \pi - \theta_A)$ fixed. Then, there exists $C > 0$, such that, for all $\eta, \mu \in \overline{S_\varphi} \subset \rho(-A)$ and all $f \in L^p(a, b; X)$ with $1 < p < +\infty$, we have

1. $$\left\| e^{-(.-a)\sqrt{-A+\eta I}} \int_a^b e^{-(s-a)\sqrt{-A+\mu I}} f(s)\ ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|f\|_{L^p(a,b;X)},$$
2. $$\left\| e^{-(.-a)\sqrt{-A+\eta I}} \int_a^b e^{-(b-s)\sqrt{-A+\mu I}} f(s)\ ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|f\|_{L^p(a,b;X)},$$
3. $$\left\| e^{-(b-.)\sqrt{-A+\eta I}} \int_a^b e^{-(b-s)\sqrt{-A+\mu I}} f(s)\ ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|f\|_{L^p(a,b;X)},$$
4. $$\left\| e^{-(b-.)\sqrt{-A+\eta I}} \int_a^b e^{-(s-a)\sqrt{-A+\mu I}} f(s)\ ds \right\|_{L^p(a,b;X)} \leqslant \left(\frac{C}{\sqrt{1+|\mu|}} + \frac{C}{\sqrt{1+|\eta|}} \right) \|f\|_{L^p(a,b;X)}.$$

Proof. We first focus on statement 1. For all $x \in [a, b]$, setting

$$v(x) = e^{-(x-a)\sqrt{-A+\eta I}} \int_a^b e^{-(s-a)\sqrt{-A+\mu I}} f(s)\ ds,$$

we have

$$\begin{aligned}
v(x) &= e^{-(x-a)\sqrt{-A+\eta I}}\int_a^x e^{-(s-a)\sqrt{-A+\mu I}}f(s)\ ds \\
&\quad + e^{-(x-a)\sqrt{-A+\eta I}}\int_x^b e^{-(s-a)\sqrt{-A+\mu I}}f(s)\ ds \\
&= \int_a^x e^{-(x-s)\sqrt{-A+\eta I}}e^{-(s-a)\left(\sqrt{-A+\mu I}+\sqrt{-A+\eta I}\right)}f(s)\ ds \\
&\quad + e^{-(x-a)\sqrt{-A+\eta I}}e^{-(x-a)\sqrt{-A+\mu I}}\int_x^b e^{-(s-x)\sqrt{-A+\mu I}}f(s)\ ds \\
&= I_{\eta,g}(x) + e^{-(x-a)\sqrt{-A+\eta I}}e^{-(x-a)\sqrt{-A+\mu I}}J_{\mu,f}(x),
\end{aligned}$$

where $g(s) = e^{-(s-a)\left(\sqrt{-A+\mu I}+\sqrt{-A+\eta I}\right)}f(s)$. Then from (38) in [14], there exists $C > 0$, independent of η and μ, such that

$$\left\| e^{-(x-a)\sqrt{-A+\eta I}}e^{-(x-a)\sqrt{-A+\mu I}} \right\|_{\mathcal{L}(X)} \leqslant C \quad \text{and} \quad \|g\|_{L^p(a,b;X)} \leqslant C\,\|f\|_{L^p(a,b;X)}.$$

Finally Lemma 4.11 gives statement 1. The other statements are obtained, in the same way.

Note that statements 1. and 2. have been proved in Lemma 5.6 in [14], in the case when $a = 0$, $b = 1$ and $\eta = \mu$. $\square$

Remark 4.13. Note that, for $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C}\setminus\overline{S_{2\theta_A}}\right)$, we have

$$M_\lambda = -\sqrt{-A_{k/2}+\mu I} \quad \text{and} \quad L_\lambda = -\sqrt{-A_{k/2}+\eta I},$$

with $\mu = -\eta = -i\sqrt{-\lambda - \dfrac{k^2}{4}}$. But, from Lemma 4.4, we deduce that $\mu, \eta \in S_{\pi-\theta_A}$, thus we can apply Lemma 4.12.

Lemma 4.14. Let $t_0 > 0$ fixed. Then

1. $\mathbb{C}\setminus\overline{S_{\theta_A}} \subset \rho(A_{k/2})$ and for any $\nu \in (0, \pi - \theta_A)$, there exists $C_\nu > 0$ such that

$$\left\| \left(-A_{k/2} - \mu I\right)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \frac{C_\nu}{|\mu|}, \quad \mu \in S_\nu.$$

2. There exists $C_{t_0} > 0$ such that, for any $\mu \in S_{\pi-\theta_A}$, we have

$$\left\| \left(I \pm e^{t_0 H_\mu}\right)^{-1} \right\|_{\mathcal{L}(X)} \leqslant C_{t_0},$$

where $H_\mu := -\sqrt{-A_{k/2} - \mu I}$.

Proof.

1. The result follows from Theorem 2, p. 437 in [31] since $-A_{k/2} \in \mathrm{BIP}\,(X, \theta_A)$.

2. We apply the same technique as in the proof of Lemma 5.1 in [14]: A is replaced by $A_{k/2}$ and e^{2A} is replaced by $e^{t_0 A_{k/2}}$.

$\square$

Due to the previous study, we can apply the results of section 3 and then obtain that, for each $i = 1, 2, 3, 4, 5$, problem (34)-(BCi)$_0$ admits a unique classical solution u_i given by (22) in Proposition 3.7, where M and L are replaced by M_λ and L_λ. We need to give estimates on u_i which take into account the dependence on λ, but u_i contains $F_{0,f}$ and $F'_{0,f}$. To this end, we give the following result.

Lemma 4.15. Let $\gamma \in \{a, b\}$. Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then, there exists $C > 0$, such that, for all $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$ and all $f \in L^p(a, b; X)$, we have

$$\begin{cases} \|v_0\|_{L^p(a,b;X)} & \leqslant \dfrac{C}{1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \|f\|_{L^p(a,b;X)} \\[2ex] \|L_\lambda v_0(.)\|_{L^p(a,b;X)} & \leqslant \dfrac{C}{\sqrt{1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}}} \|f\|_{L^p(a,b;X)} . \end{cases}$$

Moreover, for $T_\lambda = M_\lambda$ or L_λ, we have

$$\begin{cases} \left\|e^{(.-a)T_\lambda} F'_{0,f}(\gamma)\right\|_{L^p(a,b;X)} & \leqslant \dfrac{C}{\left(1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}\right)^{\frac{3}{2}}} \|f\|_{L^p(a,b;X)} \\[2ex] \left\|e^{(b-.)T_\lambda} F'_{0,f}(\gamma)\right\|_{L^p(a,b;X)} & \leqslant \dfrac{C}{\left(1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}\right)^{\frac{3}{2}}} \|f\|_{L^p(a,b;X)}, \end{cases}$$

and

$$\begin{cases} \left\|e^{(.-a)T_\lambda} L_\lambda F'_{0,f}(\gamma)\right\|_{L^p(a,b;X)} & \leqslant \dfrac{C}{1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \|f\|_{L^p(a,b;X)} \\[2ex] \left\|e^{(b-.)T_\lambda} L_\lambda F'_{0,f}(\gamma)\right\|_{L^p(a,b;X)} & \leqslant \dfrac{C}{1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \|f\|_{L^p(a,b;X)}, \end{cases}$$

where for all $x \in [a, b]$, $F'_{0,f}$ and v_0 are given by

$$\begin{aligned} F'_{0,f}(x) &= -\frac{1}{2}\left(e^{(x-a)M_\lambda} + e^{(b-x)M_\lambda} e^{cM_\lambda}\right) Z \int_a^b e^{(s-a)M_\lambda} v_0(s)\, ds \\ &\quad + \frac{1}{2}\left(e^{(b-x)M_\lambda} + e^{(x-a)M_\lambda} e^{cM_\lambda}\right) Z \int_a^b e^{(b-s)M_\lambda} v_0(s)\, ds \\ &\quad + \frac{1}{2}\int_a^x e^{(x-s)M_\lambda} v_0(s)\, ds - \frac{1}{2}\int_x^b e^{(s-x)M_\lambda} v_0(s)\, ds, \end{aligned} \tag{45}$$

and

$$\begin{aligned} v_0(x) &= \frac{1}{2}\left(e^{(b-x)L_\lambda} e^{cL_\lambda} - e^{(x-a)L_\lambda}\right) W L_\lambda^{-1} \int_a^b e^{(s-a)L_\lambda} f(s)\, ds \\ &\quad + \frac{1}{2}\left(e^{(x-a)L_\lambda} e^{cL_\lambda} - e^{(b-x)L_\lambda}\right) W L_\lambda^{-1} \int_a^b e^{(b-s)L_\lambda} f(s)\, ds \\ &\quad + \frac{1}{2} L_\lambda^{-1} \int_a^x e^{(x-s)L_\lambda} f(s)\, ds + \frac{1}{2} L_\lambda^{-1} \int_x^b e^{(s-x)L_\lambda} f(s)\, ds, \end{aligned} \tag{46}$$

with $Z = \left(I - e^{2cM_\lambda}\right)^{-1}$ and $W = \left(I - e^{2cL_\lambda}\right)^{-1}$.

Proof. In the sequel, $C > 0$ will denote various constants which are independent of f and λ.

Let $t_0 > 0$ fixed. For all $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$, we have

$$\begin{cases} M_\lambda &= -\sqrt{-A_{k/2} - i\sqrt{-\lambda - \frac{k^2}{4}}I}, \qquad \text{with} \quad i\sqrt{-\lambda - \frac{k^2}{4}} \in S_{\pi-\theta_A} \\ L_\lambda &= -\sqrt{-A_{k/2} - \left(-i\sqrt{-\lambda - \frac{k^2}{4}}I\right)}, \quad \text{with} \quad -i\sqrt{-\lambda - \frac{k^2}{4}} \in S_{\pi-\theta_A}. \end{cases}$$

Then, from Lemma 4.14, there exists $C_{t_0} > 0$ such that for all $\lambda \in -\frac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$, we obtain

$$\left\|\left(I \pm e^{t_0 M_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)} \leqslant C_{t_0} \quad \text{and} \quad \left\|\left(I \pm e^{t_0 L_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)} \leqslant C_{t_0}. \tag{47}$$

Moreover, since $-M_\lambda \in \mathrm{BIP}(X, \theta_M)$ and $-L_\lambda \in \mathrm{BIP}(X, \theta_L)$, with $\theta_M, \theta_L \in [0, \pi/2)$, from Lemma 4.8, we deduce that

$$\left\|e^{cM_\lambda}\right\|_{\mathcal{L}(X)} \leqslant C \quad \text{and} \quad \left\|e^{cL_\lambda}\right\|_{\mathcal{L}(X)} \leqslant C. \tag{48}$$

Furthermore, from [10], Lemma 2.6, statement a), p. 104, we have

$$\left\|M_\lambda^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{\sqrt{1 + \left|-i\sqrt{-\lambda - \frac{k^2}{4}}\right|}} = \frac{C}{\sqrt{1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}}}, \tag{49}$$

and

$$\left\|L_\lambda^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{\sqrt{1 + \left|i\sqrt{-\lambda - \frac{k^2}{4}}\right|}} = \frac{C}{\sqrt{1 + \sqrt{\left|\lambda + \frac{k^2}{4}\right|}}}. \tag{50}$$

From (46), since $v_0(x) \in D(L_\lambda)$, for $x \in [a, b]$, we have

$$\begin{aligned} L_\lambda v_0(x) &= \frac{1}{2} W e^{cL_\lambda} e^{(b-x)L_\lambda} \int_a^b e^{(s-a)L_\lambda} f(s)\, ds - \frac{1}{2} W e^{(x-a)L_\lambda} \int_a^b e^{(s-a)L_\lambda} f(s)\, ds \\ &+ \frac{1}{2} W e^{cL_\lambda} e^{(x-a)L_\lambda} \int_a^b e^{(b-s)L_\lambda} f(s)\, ds - \frac{1}{2} W e^{(b-x)L_\lambda} \int_a^b e^{(b-s)L_\lambda} f(s)\, ds \\ &+ \frac{1}{2} \int_a^x e^{(x-s)L_\lambda} f(s)\, ds + \frac{1}{2} \int_x^b e^{(s-x)L_\lambda} f(s)\, ds. \end{aligned}$$

Hence

$$\begin{aligned} \|L_\lambda v_0(.)\|_{L^p(a,b;X)} &\leqslant \frac{1}{2} \|W\|_{\mathcal{L}(X)} \left\|e^{cL_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(b-.)L_\lambda} \int_a^b e^{(s-a)L_\lambda} f(s)\, ds\right\|_{L^p(a,b;X)} \\ &+ \frac{1}{2} \|W\|_{\mathcal{L}(X)} \left\|e^{(.-a)L_\lambda} \int_a^b e^{(s-a)L_\lambda} f(s)\, ds\right\|_{L^p(a,b;X)} \\ &+ \frac{1}{2} \|W\|_{\mathcal{L}(X)} \left\|e^{cL_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)L_\lambda} \int_a^b e^{(b-s)L_\lambda} f(s)\, ds\right\|_{L^p(a,b;X)} \\ &+ \frac{1}{2} \|W\|_{\mathcal{L}(X)} \left\|e^{(b-.)L_\lambda} \int_a^b e^{(b-s)L_\lambda} f(s)\, ds\right\|_{L^p(a,b;X)} \\ &+ \frac{1}{2} \left\|\int_a^{.} e^{(.-s)L_\lambda} f(s)\, ds\right\|_{L^p(a,b;X)} + \frac{1}{2} \left\|\int_{.}^b e^{(s-.)L_\lambda} f(s)\, ds\right\|_{L^p(a,b;X)}, \end{aligned}$$

and from (47), (48), Lemma 4.11, Lemma 4.12 and Remark 4.13, we have

$$\|L_\lambda v_0(.)\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{1+\sqrt{\left|\lambda+\frac{k^2}{4}\right|}}} \|f\|_{L^p(a,b;X)}. \tag{51}$$

Moreover, since $v_0 = L_\lambda^{-1} L_\lambda v_0(.)$, it follows that

$$\|v_0\|_{L^p(a,b;X)} \leqslant \left\|L_\lambda^{-1}\right\|_{\mathcal{L}(X)} \|L_\lambda v_0(.)\|_{L^p(a,b;X)}.$$

Thus, from (50) and (51), it follows that

$$\|v_0\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\sqrt{\left|\lambda+\frac{k^2}{4}\right|}} \|f\|_{L^p(a,b;X)}. \tag{52}$$

In the same way, from (45), we have

$$\begin{aligned}
&L_\lambda F'_{0,f}(\gamma) = \\
&-\frac{1}{2} Z e^{(\gamma-a)M_\lambda} L_\lambda \int_a^b e^{(s-a)M_\lambda} v_0(s)\, ds - \frac{1}{2} Z e^{cM_\lambda} e^{(b-\gamma)M_\lambda} L_\lambda \int_a^b e^{(s-a)M_\lambda} v_0(s)\, ds \\
&+\frac{1}{2} Z e^{(b-\gamma)M_\lambda} L_\lambda \int_a^b e^{(b-s)M_\lambda} v_0(s)\, ds + \frac{1}{2} Z e^{cM_\lambda} e^{(\gamma-a)M_\lambda} L_\lambda \int_a^b e^{(b-s)M_\lambda} v_0(s)\, ds \\
&+\frac{1}{2} L_\lambda \int_a^\gamma e^{(\gamma-s)M_\lambda} v_0(s)\, ds - \frac{1}{2} L_\lambda \int_\gamma^b e^{(s-\gamma)M_\lambda} v_0(s)\, ds,
\end{aligned}$$

hence

$$\begin{aligned}
&L_\lambda F'_{0,f}(\gamma) = \\
&-\frac{1}{2} Z e^{(\gamma-a)M_\lambda} \int_a^b e^{(s-a)M_\lambda} L_\lambda v_0(s)\, ds - \frac{1}{2} Z e^{cM_\lambda} e^{(b-\gamma)M_\lambda} \int_a^b e^{(s-a)M_\lambda} L_\lambda v_0(s)\, ds \\
&+\frac{1}{2} Z e^{(b-\gamma)M_\lambda} \int_a^b e^{(b-s)M_\lambda} L_\lambda v_0(s)\, ds + \frac{1}{2} Z e^{cM_\lambda} e^{(\gamma-a)M_\lambda} \int_a^b e^{(b-s)M_\lambda} L_\lambda v_0(s)\, ds \\
&+\frac{1}{2} \int_a^\gamma e^{(\gamma-s)M_\lambda} L_\lambda v_0(s)\, ds - \frac{1}{2} \int_\gamma^b e^{(s-\gamma)M_\lambda} L_\lambda v_0(s)\, ds.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\left\|e^{(.-a)T_\lambda} L_\lambda F'_{0,f}(\gamma)\right\|_{L^p(a,b;X)} \leqslant \\
&\frac{1}{2} \|Z\|_{\mathcal{L}(X)} \left\|e^{(\gamma-a)M_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda} \int_a^b e^{(s-a)M_\lambda} L_\lambda v_0(s)\, ds\right\|_{L^p(a,b;X)} \\
&+\frac{1}{2} \|Z\|_{\mathcal{L}(X)} \left\|e^{cM_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(b-\gamma)M_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda} \int_a^b e^{(s-a)M_\lambda} L_\lambda v_0(s)\, ds\right\|_{L^p(a,b;X)} \\
&+\frac{1}{2} \|Z\|_{\mathcal{L}(X)} \left\|e^{(b-\gamma)M_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda} \int_a^b e^{(b-s)M_\lambda} L_\lambda v_0(s)\, ds\right\|_{L^p(a,b;X)} \\
&+\frac{1}{2} \|Z\|_{\mathcal{L}(X)} \left\|e^{cM_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(\gamma-a)M_\lambda}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda} \int_a^b e^{(b-s)M_\lambda} L_\lambda v_0(s)\, ds\right\|_{L^p(a,b;X)} \\
&+\frac{1}{2} \left\|e^{(.-a)T_\lambda} \int_a^\gamma e^{(\gamma-s)M_\lambda} L_\lambda v_0(s)\, ds\right\|_{L^p(a,b;X)} + \frac{1}{2} \left\|e^{(.-a)T_\lambda} \int_\gamma^b e^{(s-\gamma)M_\lambda} L_\lambda v_0(s)\, ds\right\|_{L^p(a,b;X)}.
\end{aligned}$$

Since $L_\lambda v_0(.) \in L^p(a,b;X)$, from (47), (48), Lemma 4.12 and Remark 4.13, we have

$$\left\| e^{(.-a)T_\lambda} L_\lambda F'_{0,f}(\gamma) \right\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{1+\sqrt{\left|\lambda + \frac{k^2}{4}\right|}}} \left\| L_\lambda v_0(.) \right\|_{L^p(a,b;X)}.$$

Moreover, from (50), we have

$$\begin{aligned} \left\| e^{(.-a)T_\lambda} F'_{0,f}(\gamma) \right\|_{L^p(a,b;X)} &= \left\| e^{(.-a)T_\lambda} L_\lambda L_\lambda^{-1} F'_{0,f}(\gamma) \right\|_{L^p(a,b;X)} \\ &\leqslant \left\| L_\lambda^{-1} \right\|_{\mathcal{L}(X)} \left\| e^{(.-a)T_\lambda} L_\lambda F'_{0,f}(\gamma) \right\|_{L^p(a,b;X)} \\ &\leqslant \frac{C}{1+\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\| L_\lambda v_0(.) \right\|_{L^p(a,b;X)}. \end{aligned}$$

In the same way, we obtain that

$$\left\| e^{(b-.)T_\lambda} L_\lambda F'_{0,f}(\gamma) \right\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{1+\sqrt{\left|\lambda + \frac{k^2}{4}\right|}}} \left\| L_\lambda v_0(.) \right\|_{L^p(a,b;X)},$$

and

$$\left\| e^{(b-.)T_\lambda} F'_{0,f}(\gamma) \right\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\| L_\lambda v_0(.) \right\|_{L^p(a,b;X)}.$$

Finally, from (51), we obtain the expected results. □

Proposition 4.16. Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then

1. for $i = 1,2,5$, there exists constants $C_i > 0$, such that, for all $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$ and all $f \in L^p(a,b;X)$, we have

$$\left\| (-\mathcal{A}_i - \lambda I)^{-1} f \right\|_{\mathcal{L}(X)} \leqslant \frac{C_i}{1+\left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)},$$

2. for $i = 3,4$, there exists $C_i > 0$, such that, for all $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \left(\overline{B(0,r)} \cup \overline{S_{2\theta_A}}\right)\right)$ and all $f \in L^p(a,b;X)$, we have

$$\left\| (-\mathcal{A}_i - \lambda I)^{-1} f \right\|_{\mathcal{L}(X)} \leqslant \frac{C_i}{1+\left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}.$$

Proof. In the sequel, $C > 0$ will denote various constants which are independent of f and λ. Moreover, we will use that, for $i = 1,2,3,4,5$, we have $(-\mathcal{A}_i - \lambda I)^{-1} f = u_i$, where u_i is the unique classical solution of (34)-(BCi)$_0$.

1. Let $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$.

   We first focus on $-\mathcal{A}_1$. Our aim is to obtain that

$$\left\| (-\mathcal{A}_1 - \lambda I)^{-1} f \right\|_{\mathcal{L}(X)} = \|u_1\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}.$$

To this end, we first recall, from Theorem 3.1, the expression of $u_1 = F_{0,f}$, given, for all $x \in [a,b]$, by

$$\begin{aligned} F_{0,f}(x) &= \frac{1}{2}\left(e^{(b-x)M_\lambda}e^{cM_\lambda} - e^{(x-a)M_\lambda}\right) ZM_\lambda^{-1}\int_a^b e^{(s-a)M_\lambda}v_0(s)\, ds \\ &+\frac{1}{2}\left(e^{(x-a)M_\lambda}e^{cM_\lambda} - e^{(b-x)M_\lambda}\right) ZM_\lambda^{-1}\int_a^b e^{(b-s)M_\lambda}v_0(s)\, ds \\ &+\frac{1}{2}M_\lambda^{-1}\int_a^x e^{(x-s)M_\lambda}v_0(s)\, ds + \frac{1}{2}M_\lambda^{-1}\int_x^b e^{(s-x)M_\lambda}v_0(s)\, ds, \end{aligned} \tag{53}$$

where v_0 is given by (46) and $Z = \left(I - e^{2cM_\lambda}\right)^{-1}$.

Moreover, following the same step as in the proof of Lemma 4.15, we have

$$\|F_{0,f}\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\sqrt{\left|\lambda + \frac{k^2}{4}\right|}}\|v_0\|_{L^p(a,b;X)},$$

and from Lemma 4.15, we have

$$\|v_0\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\sqrt{\left|\lambda + \frac{k^2}{4}\right|}}\|f\|_{L^p(a,b;X)}.$$

Then, we obtain

$$\begin{aligned} \|u_1\|_{L^p(a,b;X)} = \|F_{0,f}\|_{L^p(a,b;X)} &\leqslant \frac{C}{1+2\sqrt{\left|\lambda + \frac{k^2}{4}\right|}+\left|\lambda + \frac{k^2}{4}\right|}\|f\|_{L^p(a,b;X)} \\ &\leqslant \frac{C}{1+\left|\lambda + \frac{k^2}{4}\right|}\|f\|_{L^p(a,b;X)}. \end{aligned} \tag{54}$$

From Theorem 3.1, since $u_5 = u_1$, the result follows for u_5.

Our aim is now to show that

$$\left\|(-\mathcal{A}_2 - \lambda I)^{-1}f\right\|_{\mathcal{L}(X)} = \|u_2\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\left|\lambda + \frac{k^2}{4}\right|}\|f\|_{L^p(a,b;X)}.$$

We first recall, from (27), (28) and (29), that u_2 is given, for all $x \in [a,b]$, by

$$\begin{aligned} u_2(x) &= \left(e^{(x-a)M_\lambda} - e^{(b-x)M_\lambda}\right)\alpha_1 + \left(e^{(x-a)L_\lambda} - e^{(b-x)L_\lambda}\right)\alpha_2 \\ &+ \left(e^{(x-a)M_\lambda} + e^{(b-x)M_\lambda}\right)\alpha_3 + \left(e^{(x-a)L_\lambda} + e^{(b-x)L_\lambda}\right)\alpha_4 + F_{0,f}(x), \end{aligned}$$

where $F_{0,f} = u_1$ is given by (53) and

$$\left\{\begin{aligned} \alpha_1 &= -\frac{1}{2}\left(I + e^{cM_\lambda}\right)^{-1} M_\lambda^{-1}\left(F'_{0,f}(a) + F'_{0,f}(b)\right) \\ \alpha_2 &= 0 \\ \alpha_3 &= -\frac{1}{2}\left(I - e^{cM_\lambda}\right)^{-1} M_\lambda^{-1}\left(F'_{0,f}(a) - F'_{0,f}(b)\right) \\ \alpha_4 &= 0. \end{aligned}\right.$$

Thus, we have

$$\begin{aligned}
\|u_2\|_{L^p(a,b;X)} \leqslant\ & \frac{1}{2}\left\|\left(I+e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(.-a)M_\lambda}F'_{0,f}(a)\right\|_{L^p(a,b;X)}\\
&+\frac{1}{2}\left\|\left(I+e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(.-a)M_\lambda}F'_{0,f}(b)\right\|_{L^p(a,b;X)}\\
&+\frac{1}{2}\left\|\left(I+e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(b-.)M_\lambda}F'_{0,f}(a)\right\|_{L^p(a,b;X)}\\
&+\frac{1}{2}\left\|\left(I+e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(b-.)M_\lambda}F'_{0,f}(b)\right\|_{L^p(a,b;X)}\\
&+\frac{1}{2}\left\|\left(I-e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(.-a)M_\lambda}F'_{0,f}(a)\right\|_{L^p(a,b;X)}\\
&+\frac{1}{2}\left\|\left(I-e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(.-a)M_\lambda}F'_{0,f}(b)\right\|_{L^p(a,b;X)}\\
&+\frac{1}{2}\left\|\left(I-e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(b-.)M_\lambda}F'_{0,f}(a)\right\|_{L^p(a,b;X)}\\
&+\frac{1}{2}\left\|\left(I-e^{cM_\lambda}\right)^{-1}\right\|_{\mathcal{L}(X)}\|M_\lambda^{-1}\|_{\mathcal{L}(X)}\left\|e^{(b-.)M_\lambda}F'_{0,f}(b)\right\|_{L^p(a,b;X)}\\
&+\|F_{0,f}\|_{L^p(a,b;X)},
\end{aligned}$$

and from (47), (49), Lemma 4.15 and (54), we have

$$\|u_2\|_{L^p(a,b;X)} \leqslant \frac{C}{1+2\sqrt{\left|\lambda+\frac{k^2}{4}\right|}+\left|\lambda+\frac{k^2}{4}\right|}\|f\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\left|\lambda+\frac{k^2}{4}\right|}\|f\|_{L^p(a,b;X)}.$$

2. Let $\lambda \in -\dfrac{k^2}{4}+\left(\mathbb{C}\setminus\left(\overline{B(0,r)}\cup\overline{S_{2\theta_A}}\right)\right)$.
   Our aim is to obtain that

$$\left\|(-\mathcal{A}_i-\lambda I)^{-1}f\right\|_{\mathcal{L}(X)} = \|u_i\|_{L^p(a,b;X)} \leqslant \frac{C}{1+\left|\lambda+\frac{k^2}{4}\right|}\|f\|_{L^p(a,b;X)}, \quad \text{for } i=3,4.$$

Recall, from (27), (31) and (30), that u_3 is given by

$$\begin{aligned}
u_3(x) =\ & \left(e^{(x-a)M_\lambda}-e^{(b-x)M_\lambda}\right)\alpha_1+\left(e^{(x-a)L_\lambda}-e^{(b-x)L_\lambda}\right)\alpha_2\\
&+\left(e^{(x-a)M_\lambda}+e^{(b-x)M_\lambda}\right)\alpha_3+\left(e^{(x-a)L_\lambda}+e^{(b-x)L_\lambda}\right)\alpha_4+F_{0,f}(x),
\end{aligned}\tag{55}$$

where $F_{0,f}=u_1$ is given by (53) and

$$\left\{\begin{aligned}
\alpha_1 &= \frac{1}{2}B_\lambda^{-1}(L_\lambda+M_\lambda)U_\lambda^{-1}\left(I-e^{cL_\lambda}\right)\left(F'_{0,f}(a)+F'_{0,f}(b)\right)\\
\alpha_2 &= -\frac{1}{2}B_\lambda^{-1}(L_\lambda+M_\lambda)U_\lambda^{-1}\left(I-e^{cM_\lambda}\right)\left(F'_{0,f}(a)+F'_{0,f}(b)\right)\\
\alpha_3 &= \frac{1}{2}B_\lambda^{-1}(L_\lambda+M_\lambda)V_\lambda^{-1}\left(I+e^{cL_\lambda}\right)\left(F'_{0,f}(a)-F'_{0,f}(b)\right)\\
\alpha_4 &= -\frac{1}{2}B_\lambda^{-1}(L_\lambda+M_\lambda)V_\lambda^{-1}\left(I+e^{cM_\lambda}\right)\left(F'_{0,f}(a)-F'_{0,f}(b)\right).
\end{aligned}\right.$$

Moreover, let

$$T_\lambda = M_\lambda \text{ or } L_\lambda,$$

then from (42), (44), (47), Lemma 4.7 and Lemma 4.15, for $i = 1, 2, 3, 4$, we have

$$\begin{aligned}
\|e^{(.-a)T_\lambda}\alpha_i\|_{L^p(a,b;X)} &\leqslant \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\|e^{(.-a)T_\lambda}(L_\lambda + M_\lambda)F'_{0,f}(a)\right\|_{L^p(a,b;X)} \\
&\quad + \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\|e^{(.-a)T_\lambda}(L_\lambda + M_\lambda)F'_{0,f}(b)\right\|_{L^p(a,b;X)} \\
&\leqslant \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\|(L_\lambda + M_\lambda)L_\lambda^{-1}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(a)\right\|_{L^p(a,b;X)} \\
&\quad + \frac{C}{\sqrt{|\lambda + \frac{k^2}{4}|}} \left\|(L_\lambda + M_\lambda)L_\lambda^{-1}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(b)\right\|_{L^p(a,b;X)} \\
&\leqslant \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\|I + M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(a)\right\|_{L^p(a,b;X)} \\
&\quad + \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\|I + M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)} \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(b)\right\|_{L^p(a,b;X)},
\end{aligned}$$

hence

$$\begin{aligned}
\|e^{(.-a)T_\lambda}\alpha_i\|_{L^p(a,b;X)} &\leqslant \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left(1 + \left\|M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)}\right) \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(a)\right\|_{L^p(a,b;X)} \\
&\quad + \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left(1 + \left\|M_\lambda L_\lambda^{-1}\right\|_{\mathcal{L}(X)}\right) \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(b)\right\|_{L^p(a,b;X)} \\
&\leqslant \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(a)\right\|_{L^p(a,b;X)} \\
&\quad + \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|}} \left\|e^{(.-a)T_\lambda}L_\lambda F'_{0,f}(b)\right\|_{L^p(a,b;X)} \\
&\leqslant \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|} + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}.
\end{aligned}$$

In the same way, we obtain that

$$\left\|e^{(b-.)T_\lambda}\alpha_i\right\|_{L^p(a,b;X)} \leqslant \frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|} + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}.$$

Moreover, due to (43), we have

$$\sqrt{\left|\lambda + \frac{k^2}{4}\right|} \geqslant \sqrt{r} > 0.$$

Hence

$$\frac{C}{\sqrt{\left|\lambda + \frac{k^2}{4}\right|} + \left|\lambda + \frac{k^2}{4}\right|} \leqslant \frac{C}{\sqrt{r} + \left|\lambda + \frac{k^2}{4}\right|}.$$

Therefore, for $i = 1, 2, 3, 4$, we have

$$\left\| e^{(.-a)T_\lambda} \alpha_i \right\|_{L^p(a,b;X)} \leqslant \frac{C}{1 + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}, \tag{56}$$

and

$$\left\| e^{(b-.)T_\lambda} \alpha_i \right\|_{L^p(a,b;X)} \leqslant \frac{C}{1 + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}, \tag{57}$$

Finally, from (54), (55), (56) and (57), we obtain

$$\|u_3\|_{L^p(a,b;X)} \leqslant \frac{C}{1 + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}.$$

Now, we focus ourselves on $-\mathcal{A}_4$. Recall that, from (27), (29) and (32), u_4 is given, for all $x \in [a, b]$, by

$$\begin{aligned} u_4(x) \quad = \quad & \left(e^{(x-a)M_\lambda} - e^{(b-x)M_\lambda}\right) \alpha_1 + \left(e^{(x-a)L_\lambda} - e^{(b-x)L_\lambda}\right) \alpha_2 \\ & + \left(e^{(x-a)M_\lambda} + e^{(b-x)M_\lambda}\right) \alpha_3 + \left(e^{(x-a)L_\lambda} + e^{(b-x)L_\lambda}\right) \alpha_4 + F_{0,f}(x), \end{aligned} \tag{58}$$

where $F_{0,f} = u_1$ is given by (53) and

$$\left\{ \begin{array}{lcl} \alpha_1 & = & -\dfrac{1}{2} B_\lambda^{-1} (L_\lambda + M_\lambda) V_\lambda^{-1} \left(I - e^{cL_\lambda}\right) L_\lambda M_\lambda^{-1} \left(F'_{0,f}(a) + F'_{0,f}(b)\right) \\ \alpha_2 & = & \dfrac{1}{2} B_\lambda^{-1} (L_\lambda + M_\lambda) V_\lambda^{-1} \left(I - e^{cM_\lambda}\right) M_\lambda L_\lambda^{-1} \left(F'_{0,f}(a) + F'_{0,f}(b)\right) \\ \alpha_3 & = & -\dfrac{1}{2} B_\lambda^{-1} (L_\lambda + M_\lambda) U_\lambda^{-1} \left(I + e^{cL_\lambda}\right) L_\lambda M_\lambda^{-1} \left(F'_{0,f}(a) - F'_{0,f}(b)\right) \\ \alpha_4 & = & \dfrac{1}{2} B_\lambda^{-1} (L_\lambda + M_\lambda) U_\lambda^{-1} \left(I + e^{cM_\lambda}\right) M_\lambda L_\lambda^{-1} \left(F'_{0,f}(a) - F'_{0,f}(b)\right). \end{array} \right.$$

As previously, from (42), (43), (44), (47), Lemma 4.7 and Lemma 4.15, for $i = 1, 2, 3, 4$, we have

$$\left\| e^{(.-a)T_\lambda} \alpha_i \right\|_{L^p(a,b;X)} \leqslant \frac{C}{1 + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}, \tag{59}$$

and

$$\left\| e^{(b-.)T_\lambda} \alpha_i \right\|_{L^p(a,b;X)} \leqslant \frac{C}{1 + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}. \tag{60}$$

Finally, from (54), (58), (59) and (60), we obtain

$$\|u_4\|_{L^p(a,b;X)} \leqslant \frac{C}{1 + \left|\lambda + \frac{k^2}{4}\right|} \|f\|_{L^p(a,b;X)}.$$

$\square$

4.4 Proof of Proposition 4.1

For $i = 1, 2, 5$. From Proposition 4.9, we have

$$\sigma\left(-\mathcal{A}_i + \frac{k^2}{4} I\right) \subset \overline{S_{2\theta_A}},$$

and from Proposition 4.16, for all $\lambda \in \left(\mathbb{C} \setminus \overline{S_{2\theta_A}}\right)$, there exist $C_i \geqslant 1$, such that

$$\left\| \left(-\mathcal{A}_i + \frac{k^2}{4} I - \lambda I\right)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \frac{C_i}{1+|\lambda|}.$$

Then, due to Definition 2.2, we deduce that $-\mathcal{A}_i + \dfrac{k^2}{4} I$, is a sectorial operator of angle $2\theta_A$.

For $i = 3, 4$. From Proposition 4.10, there exists $r > 0$, such that

$$\sigma\left(-\mathcal{A}_i + \frac{k^2}{4} I\right) \subset \overline{B\left(-\frac{k^2}{4}, r\right)} \cup \overline{S_{2\theta_A}}. \tag{61}$$

Thus

$$\begin{cases} \sigma\left(-\mathcal{A}_i + \dfrac{k^2}{4} I + rI\right) \subset \overline{S_{\frac{\pi}{2}}}, & \text{if } 2\theta_A < \frac{\pi}{2} \\ \sigma\left(-\mathcal{A}_i + \dfrac{k^2}{4} I + rI\right) \subset \overline{S_{2\theta_A}}, & \text{if } 2\theta_A \geqslant \frac{\pi}{2}. \end{cases}$$

Moreover, due to Proposition 4.16, there exist constants $C_i \geqslant 1$, such that for all complex numbers $\lambda \in -\dfrac{k^2}{4} + \left(\mathbb{C} \setminus \left(\overline{B(0, r)} \cup \overline{S_{2\theta_A}}\right)\right)$, we have

$$\left\| \left(-\mathcal{A}_i + \frac{k^2}{4} I + rI - \lambda I\right)^{-1} \right\|_{\mathcal{L}(X)} \leqslant \frac{C_i}{1+|\lambda|}.$$

Then, due to Definition 2.2, we deduce that $-\mathcal{A}_i + \dfrac{k^2}{4} I + rI$, is a sectorial operator of angle $\dfrac{\pi}{2}$ if $2\theta_A < \dfrac{\pi}{2}$ or of angle $2\theta_A$ if $2\theta_A \geqslant \dfrac{\pi}{2}$.

Moreover, when $2\theta_A \in \left(0, \dfrac{\pi}{2}\right)$, it is clear from (61) that, there exists $r' > r$ large enough, such that

$$\sigma\left(-\mathcal{A}_i + \frac{k^2}{4} I + r' I\right) \subset \overline{S_{2\theta_A}},$$

and in the same way, if $\theta_A = 0$, there exists $\theta_0 > 0$ such that

$$\sigma\left(-\mathcal{A}_i + \frac{k^2}{4} I + r' I\right) \subset \overline{S_{\theta_0}},$$

which gives the result.

5 Application

Let $T > 0$ and $i = 1, 2, 3, 4, 5$. Recall problem (2):

$$\begin{cases} v'(t) - \mathcal{A}_i v(t) = f(t), & t \in (0, T] \\ v(0) = v_0, \end{cases} \tag{62}$$

which is set in the space

$$\mathcal{E} := L^p(a, b; X).$$

Here $\mathcal{A}_i$ is defined by (33). We assume that

$$
\begin{aligned}
&(\mathcal{H}_1) \quad X \text{ is a UMD space,}\\
&(\mathcal{H}_2) \quad 0 \in \rho(A),\\
&(\mathcal{H}_3) \quad -A \in \mathrm{BIP}(X, \theta_A), \quad \text{for } \theta_A \in [0, \pi/4),\\
&(\mathcal{H}_4) \quad [k, +\infty) \in \rho(A).
\end{aligned}
$$

Thus, due to Theorem 4.2, $\mathcal{A}_i$ is the infinitesimal generator of a strongly continuous analytic semigroup in $\mathcal{E}$. Note that if $k \geqslant 0$, $(\mathcal{H}_4)$ follows from $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.

In the sequel, we will consider two cases:

1. $f \in W^{\theta,p}(0, T; \mathcal{E})$, $\theta \in \left(0, \frac{1}{p}\right)$,
2. $f \in C^{\theta}([0, T]; \mathcal{E})$, $\theta \in (0, 1)$.

5.1 First case

Assume that $f \in W^{\theta,p}(0, T; \mathcal{E})$ with $\theta \in \left(0, \frac{1}{p}\right)$ (see [7], p. 330, for the definition of such a space). From [7], Theorem 4.7, p. 334, there exists a unique classical solution u of problem (62) if and only if $v_0 \in (D(\mathcal{A}_i), \mathcal{E})_{\frac{1}{p},p}$.

5.2 Second case

Now, assume that $f \in C^{\theta}([0, T]; \mathcal{E})$, $\theta \in (0, 1)$ and $v_0 \in D(\mathcal{A}_i)$. We then apply Theorem 4.5, p. 53 in [33] to obtain the existence and the uniqueness of a solution $v \in C([0, T]; \mathcal{E})$ to problem (62) such that

$$
v' \in C^{\theta}([0, T]; \mathcal{E}) \quad \text{and} \quad \mathcal{A}_i v(.) \in C^{\theta}([0, T]; \mathcal{E}),
$$

if and only if

$$
f(0) + \mathcal{A}_i v_0 \in (D(\mathcal{A}_i), \mathcal{E})_{1-\theta,+\infty}.
$$

Remark that in this case, $\overline{D(\mathcal{A}_i)} = \mathcal{E}$.

Remark 5.1.

1. We can prove easily that operators $-\mathcal{A}_1$ and $-\mathcal{A}_5$ are BIP $(\mathcal{E}, 2\theta_A + \varepsilon)$, $\varepsilon > 0$. Therefore, using the Dore-Venni Theorem, we can solve (62) for $f \in L^p(0, T; \mathcal{E})$.
2. We can explicit all the above results in the concrete case, that is:

$$
A = A_0 \quad \text{and} \quad \mathcal{E} = L^p(a, b; L^p(\omega)).
$$

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