A biharmonic transmission problem in L^p-spaces

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Abstract

In this work we study, by a semigroup approach, a transmission problem based on biharmonic equations with boundary and transmission conditions, in two juxtaposed habitats. We give a result of existence and uniqueness of the classical solution in L^p-spaces, for $p \in (1, +\infty)$, using analytic semigroups and operators sum theory in Banach spaces. To this end, we invert explicitly the determinant operator of the transmission system in L^p-spaces using the $\mathcal{E}_\infty$-calculus and the Dore-Venni sums theory.

Key Words and Phrases: Analytic semigroups, biharmonic equations, functional calculus, interpolation spaces, maximal regularity.

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1 Introduction

In this work, we consider a system of linear biharmonic equations posed on two juxtaposed domains and coupled through transmission conditions at the interface. Throughout the paper we shall impose the continuity of the flux, of the dispersal and of its flux across the interface. Using an operator approach, we investigate the existence, uniqueness as well as maximal L^p-regularity for such a problem.

Transmission problems arise in various applicative fields including engineering, physics and biology. Here, we refer the reader for instance to [1], [9] or [15] for applications in plate theory, to [18], [26] or [34] for applications in electromagnetism and to [10], [21] or [33] for other applications in population dynamics. Let us also mention that mathematical models involving biharmonic operators also arise in various fields such as elasticity for instance see [6], [16] or [30], electrostatic see [4], [13] or [22], plate theory [1], [9] or [15] or population dynamics [5], [19], [20] or [25].

In this work, we consider an n-dimensional (with $n \geqslant 2$) straight cylinder of the form

$$\Omega = (a, b) \times \omega,$$

where $a < b$ are two given real numbers while the section $\omega \subset \mathbb{R}^{n-1}$ denotes a smooth bounded domain. This cylinder Ω is split into two (open) sub-cylinders $\Omega_\pm$ and an interface Γ given for some $\gamma \in (a, b)$ by

$$\Omega_- = (a, \gamma) \times \omega, \quad \Omega_+ = (\gamma, b) \times \omega \quad \text{and} \quad \Gamma = \{\gamma\} \times \omega,$$

so that $\Omega = \Omega_- \cup \Gamma \cup \Omega_+$.

We consider the following biharmonic equations,

$$(EQ_{pde}) \left\{ \begin{array}{lll} k_- \Delta^2 u_- = g_- & \text{in} & \Omega_- \\ k_+ \Delta^2 u_+ = g_+ & \text{in} & \Omega_+, \end{array} \right.$$

where $g_- \in L^p(\Omega_-)$, $g_+ \in L^p(\Omega_+)$ are given and $k_+, k_- > 0$.

We denote by (x, y) the spatial variables with $x \in (a, b)$ and $y \in \omega$. Then, we consider the following conditions on $\partial\Omega \setminus \Gamma$, the lateral boundary of Ω,

$$
(BC_{pde}) \begin{cases} (1) \begin{cases} u_-(x,\zeta) = 0, & x \in (a,\gamma),\ \zeta \in \partial\omega, \quad u_+(x,\zeta) = 0, \quad x \in (\gamma,b),\ \zeta \in \partial\omega \\ \Delta u_-(x,\zeta) = 0, & x \in (a,\gamma),\ \zeta \in \partial\omega, \quad \Delta u_+(x,\zeta) = 0, \quad x \in (\gamma,b),\ \zeta \in \partial\omega \end{cases} \\ (2) \begin{cases} u_-(a,y) = \varphi_1^-(y), & u_+(b,y) = \varphi_1^+(y), \quad y \in \omega \\ \dfrac{\partial u_-}{\partial x}(a,y) = \varphi_2^-(y), & \dfrac{\partial u_+}{\partial x}(b,y) = \varphi_2^+(y), \quad y \in \omega, \end{cases} \end{cases}
$$

where $\varphi_1^\pm$ and $\varphi_2^\pm$ will be given in appropriated spaces. The system is coupled on the interface Γ where we impose the following continuity conditions,

$$
(TC_{pde}) \begin{cases} u_- = u_+ & \text{on } \Gamma \\ \dfrac{\partial u_-}{\partial x} = \dfrac{\partial u_+}{\partial x} & \text{on } \Gamma \\ k_- \Delta u_- = k_+ \Delta u_+ & \text{on } \Gamma \\ k_- \dfrac{\partial \Delta u_-}{\partial x} = k_+ \dfrac{\partial \Delta u_+}{\partial x} & \text{on } \Gamma. \end{cases}
$$

Let us now explain, for instance, in population dynamics framework, the boundary and transmission conditions.

The first line of $(BC_{pde}) - (1)$, means that the individuals could not lie on the boundaries $(a, b) \times \partial\omega$, because, for instance, they die or the edge is impassable. The second line means that there is no dispersal in the normal direction. We deduce that the dispersal vanishes on $(a, b) \times \partial\omega$.

In $(BC_{pde}) - (2)$, the population density and the flux are given on $\{a, b\} \times \omega$. This means that the habitats are not isolated.

In (TC_{pde}), the two first transmission conditions mean the continuity of the density and its flux at the interface, while the two second express, in some sense, the continuity of the dispersal and its flux at Γ.

This work is a natural continuation of that done in [31]. Moreover, it completes the study realized in [20] where the authors have considered equations

$$
k_\pm \Delta^2 u_\pm - l_\pm \Delta u_\pm = f_\pm \quad \text{in} \quad \Omega_\pm
$$

and on the interface Γ, the last condition of (TC_{pde}) is replaced by

$$
\frac{\partial}{\partial x}\left(k_+ \Delta u_+ - l_+ u_+\right) \quad \text{on} \quad \Gamma.
$$

In the present work, $l_\pm = 0$, which must be treated differently, since the proof in [20] uses the representation formula obained in [19] which is not defined when $l_\pm = 0$. Thus, we use the representation formula obtained in [31] which is different and does not corresponds to the representation formula in [19] when the limit of $l_\pm$ tends to 0. Therefore, even if the proof follows the same steps than the one in [20], the calculus are different and cannot be deduce from those in [20], since they use a different representation formula.

Note that, due to the transmission conditions, we can not obtain a solution $u \in W^{4,p}(\Omega)$ in all Ω. We only can obtain a solution u such that $u_{|\Omega_-} \in W^{4,p}(\Omega_-)$ and $u_{|\Omega_+} \in W^{4,p}(\Omega_+)$.

The paper is organized as follows.

First, in section 2, we recall the PDE transmission problem (P_{pde}) and we rewrite it under operational form. Then, in section 3, we recall some definitions about BIP operator and interpolation spaces. We give our hypotheses and their consequences. We present our main result in Theorem 3.9 and Corollary 3.11 which states existence and uniqueness of the solution of problem (P_{pde}) that is $(EQ_{pde}) - (BC_{pde}) - (TC_{pde})$ quoted above. In section 4, we state technical results which allow us to prove our main result. Section 5, is devoted to the proof of Theorem 3.9.

2 Operational formulation

In this section, we first recall the PDE problem (P_{pde}) composed by $(EQ_{pde})-(BC_{pde})-(TC_{pde})$. Then, we define the Laplace operator A_0 and we use it to rewrite (P_{pde}). Note that this operational form is a vector values problem. Finally, we generalize this problem replacing A_0 by a more general operator A. We consider the following problem

$$(P_{pde})\left\{\begin{array}{llllll}
k_-\Delta^2 u_- & = & g_- & & & \text{in } \Omega_- \\
k_+\Delta^2 u_+ & = & g_+ & & & \text{in } \Omega_+ \\
u_-(x,\zeta) & = & 0, & \Delta u_-(x,\zeta) & = & 0, \quad x\in\ (a,\gamma),\ \zeta\in\partial\omega \\
u_+(x,\zeta) & = & 0, & \Delta u_+(x,\zeta) & = & 0, \quad x\in\ (\gamma,b),\ \zeta\in\partial\omega \\
u_-(a,y) & = & \varphi_1^-(y), & u_+(b,y) & = & \varphi_1^+(y), \quad y\in\omega \\
\dfrac{\partial u_-}{\partial x}(a,y) & = & \varphi_2^-(y), & \dfrac{\partial u_+}{\partial x}(b,y) & = & \varphi_2^+(y), \quad y\in\omega \\
u_- & = & u_+ & & & \text{on } \Gamma \\
\dfrac{\partial u_-}{\partial x} & = & \dfrac{\partial u_+}{\partial x} & & & \text{on } \Gamma \\
k_-\Delta u_- & = & k_+\Delta u_+ & & & \text{on } \Gamma \\
k_-\dfrac{\partial \Delta u_-}{\partial x} & = & k_+\dfrac{\partial \Delta u_+}{\partial x} & & & \text{on } \Gamma.
\end{array}\right.$$

Let us define A_0, the Dirichlet Laplace operator in $\mathbb{R}^{n-1}$, $n\in\mathbb{N}\setminus\{0,1\}$, as follows

$$\left\{\begin{array}{l}
D(A_0) := \{\psi\in W^{2,p}(\omega) : \psi = 0 \text{ on } \partial\omega\} \\
\forall\psi\in D(A_0), \quad A_0\psi = \Delta_y\psi.
\end{array}\right. \tag{1}$$

Thus, using operator A_0, problem (P_{pde}) becomes the following vector valued problem

$$\left\{\begin{array}{lll}
u_-^{(4)}(x) + 2A_0 u_-''(x) + A_0^2 u_-(x) & = & f_-(x), \quad \text{for a.e. } x\in(a,\gamma) \\
u_+^{(4)}(x) + 2A_0 u_+''(x) + A_0^2 u_+(x) & = & f_+(x), \quad \text{for a.e. } x\in(\gamma,b) \\
u_-(a) = \varphi_1^-, & & u_+(b) = \varphi_1^+ \\
u_-'(a) = \varphi_2^-, & & u_+'(b) = \varphi_2^+ \\
u_-(\gamma) = u_+(\gamma) & & \\
u_-'(\gamma) = u_+'(\gamma) & & \\
k_-u_-''(\gamma) + k_-A_0u_-(\gamma) & = & k_+u_+''(\gamma) + k_+A_0u_+(\gamma) \\
k_-u_-^{(3)}(\gamma) + k_-A_0u_-'(\gamma) & = & k_+u_+^{(3)}(\gamma) + k_+A_0u_+'(\gamma),
\end{array}\right.$$

where

$$f_-\in L^p(a,\gamma;L^p(\omega)), \quad f_+\in L^p(\gamma,b;L^p(\omega)) \quad \text{and} \quad p\in(1,+\infty),$$

with

$$u_\pm(x) := u(x,\cdot) \quad \text{and} \quad f_\pm(x) := g_\pm(x,\cdot)/k_\pm.$$

Note that the boundary conditions on $\partial\omega$ in (P_{pde}) do not appear in the previous system since they are already included in the domain of A_0. Thus, a classical solution of this problem satisfies the boundary conditions on $\partial\omega$ in (P_{pde}).

Then, using operator $(A,D(A))$ instead of $(A_0,D(A_0))$ and X instead of $L^p(\omega)$, we can write that $f_-\in L^p(a,\gamma;X)$ and $f_+\in L^p(\gamma,b;X)$.

Moreover, in all the sequel, we will study the following more general transmission problem :

$$
(\mathrm{P})\left\{\begin{array}{l}
(EQ)\left\{\begin{array}{rcll}
u_-^{(4)}(x)+2Au_-''(x)+A^2u_-(x) & = & f_-(x), & \text{for a.e. } x\in(a,\gamma)\\
u_+^{(4)}(x)+2Au_+''(x)+A^2u_+(x) & = & f_+(x), & \text{for a.e. } x\in(\gamma,b)
\end{array}\right.\\
(BC)\left\{\begin{array}{rclrcl}
u_-(a) & = & \varphi_1^-, & u_+(b) & = & \varphi_1^+\\
u_-'(a) & = & \varphi_2^-, & u_+'(b) & = & \varphi_2^+
\end{array}\right.\\
(TC)\left\{\begin{array}{rcl}
u_-(\gamma) & = & u_+(\gamma)\\
u_-'(\gamma) & = & u_+'(\gamma)\\
k_-u_-''(\gamma)\ +k_-Au_-(\gamma) & = & k_+u_+''(\gamma)\ +k_+Au_+(\gamma)\\
k_-u_-^{(3)}(\gamma)+k_-Au_-'(\gamma) & = & k_+u_+^{(3)}(\gamma)+k_+Au_+'(\gamma).
\end{array}\right.
\end{array}\right.
$$

The transmission conditions (TC) will be split into

$$
(TC1)\left\{\begin{array}{rcl}
u_-(\gamma) & = & u_+(\gamma)\\
u_-'(\gamma) & = & u_+'(\gamma),
\end{array}\right.
$$

and

$$
(TC2)\left\{\begin{array}{rcl}
k_-u_-''(\gamma)\ +k_-Au_-(\gamma) & = & k_+u_+''(\gamma)\ +k_+Au_+(\gamma)\\
k_-u_-^{(3)}(\gamma)+k_-Au_-'(\gamma) & = & k_+u_+^{(3)}(\gamma)+k_+Au_+'(\gamma).
\end{array}\right.
$$

Note that $(TC2)$ is well defined from Lemma 3.7, see Section 3.2 below.

We will search a classical solution of problem (P), that is a solution u such that

$$
\left\{\begin{array}{l}
u_-:=u_{|(a,\gamma)}\in W^{4,p}(a,\gamma;X)\cap L^p(a,\gamma;D(A^2)),\quad u_-''\in L^p(a,\gamma;D(A)),\\
u_+:=u_{|(\gamma,b)}\in W^{4,p}(\gamma,b;X)\cap L^p(\gamma,b;D(A^2)),\quad u_+''\in L^p(\gamma,b;D(A)),
\end{array}\right. \tag{2}
$$

and which satisfies $(EQ)-(BC)-(TC)$.

3 Assumptions, consequences and statement of results

3.1 The class $\mathbf{BIP}(\mathbf{X},\theta)$

Throughout the article, $(X,\|\cdot\|)$ is a complex Banach space. We give some definitions about UMD spaces, sectorial operators and BIP operators.

Definition 3.1. A Banach space X is a UMD space if and only if for all $p\in(1,+\infty)$, the Hilbert transform is bounded from $L^p(\mathbb{R},X)$ into itself (see [2] and [3]).

Definition 3.2. A closed linear operator T_1 is called sectorial of angle $\theta\in[0,\pi)$ if

$i)\quad \sigma(T_1)\subset\overline{S_\theta}$,

$ii)\quad \forall\,\theta'\in(\theta,\pi),\quad \sup\left\{\|\lambda(\lambda I-T_1)^{-1}\|_{\mathcal{L}(X)}:\ \lambda\in\mathbb{C}\setminus\overline{S_{\omega'}}\right\}<+\infty,$

where

$$
S_\omega\ :=\ \left\{\begin{array}{ll}
\{z\in\mathbb{C}: z\neq0\ \text{ and }\ |\arg(z)|<\omega\} & \text{if}\quad \omega\in(0,\pi),\\
(0,+\infty) & \text{if}\quad \omega=0,
\end{array}\right. \tag{3}
$$

see [14], p. 19, such an operator is noted $A\in\mathrm{Sect}\,(\omega)$.

Remark 3.3. From [17], p. 342, we know that any injective sectorial operator T_1 admits imaginary powers T_1^{is}, $s \in \mathbb{R}$, but, in general, T_1^{is} is not bounded.

Definition 3.4. Let $\theta \in [0, \pi)$. We denote by $\mathrm{BIP}(X, \theta)$, the class of sectorial injective operators T_1 such that

$$i) \quad \overline{D(T_1)} = \overline{R(T_1)} = X,$$

$$ii) \quad \forall\, s \in \mathbb{R}, \quad T_1^{is} \in \mathcal{L}(X),$$

$$iii) \quad \exists\, C \geq 1,\ \forall\, s \in \mathbb{R}, \quad ||T_1^{is}||_{\mathcal{L}(X)} \leq Ce^{|s|\theta},$$

see [27], p. 430.

Remark 3.5. From [14], proof of Proposition 3.2.1, c), p. 71, $\overline{D(T_1) \cap R(T_1)} = X$.

3.2 Interpolation spaces

Here we recall a definition given, for instance, in [7], [12], [23] or in [32] and a result from [12] concerning real interpolation spaces.

Definition 3.6. Let $T_2 : D(T_2) \subset X \longrightarrow X$ be a linear operator such that

$$(0, +\infty) \subset \rho(T_2) \quad \text{and} \quad \exists\, C > 0 : \forall\, t > 0, \quad \|t(T_2 - tI)^{-1}\|_{\mathcal{L}(X)} \leqslant C. \tag{4}$$

Let $k \in \mathbb{N} \setminus \{0\}$, $\theta \in (0,1)$ and $q \in [1, +\infty]$. We will use the real interpolation spaces

$$(D(T_2^k), X)_{\theta,q} = (X, D(T_2^k))_{1\text{-}\theta,q},$$

defined, for instance, in [23], or in [24].

In particular, for $k = 1$, we have the following characterization

$$(D(T_2), X)_{\theta,q} := \left\{\psi \in X : t \longmapsto t^{1\text{-}\theta}\|T_2(T_2 - tI)^{-1}\psi\|_X \in L^q_*(0, +\infty)\right\},$$

where $L^q_*(0, +\infty)$ is given by

$$L^q_*(0, +\infty) := \left\{f \in L^q(0, +\infty) : \left(\int_0^{+\infty} \|f(t)\|^q \, \frac{dt}{t}\right)^{1/q} < +\infty\right\}, \quad \text{for } q \in [1, +\infty),$$

and for $q = +\infty$, by

$$L^\infty_*(0, +\infty; \mathbb{C}) := \left\{f \text{ measurable on } (0, +\infty) : \sup_{t \in (0,+\infty)} |f(t)| < +\infty\right\},$$

see [7] p. 325, or [12], p. 665, Teorema 3, or section 1.14 of [32], where this space is denoted by $(X, D(T_2))_{1\text{-}\theta,q}$. Note that we can also characterize the space $(D(T_2), X)_{\theta,q}$ taking into account the Osservazione, p. 666, in [12].

We set also, for any $k \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned} (D(T_2), X)_{k+\theta,q} &:= \left\{\psi \in D(T_2^k) : T_2^k \psi \in (D(T_2), X)_{\theta,q}\right\}, \\ (X, D(T_2))_{k+\theta,q} &:= \left\{\psi \in D(T_2^k) : T_2^k \psi \in (X, D(T_2))_{\theta,q}\right\}. \end{aligned}$$

We recall the following lemma.

Lemma 3.7 ([12]). Let T_2 be a linear operator satisfying (4). Let u such that

$$u \in W^{n,p}(a_1, b_1; X) \cap L^p(a_1, b_1; D(T_2^k)),$$

where $a_1, b_1 \in \mathbb{R}$ with $a_1 < b_1$, $n, k \in \mathbb{N} \setminus \{0\}$ and $p \in (1, +\infty)$. Then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $s \in \{a_1, b_1\}$, we have

$$u^{(j)}(s) \in (D(T_2^k), X)_{\frac{j}{n}+\frac{1}{np},p}.$$

This result is proved in [12], p. 678, Teorema 2'.

3.3 Hypotheses

Througout this article, $k_+, k_- \in \mathbb{R}_+ \setminus \{0\}$ and A denotes a closed linear operator in X.

We assume the following hypotheses:

$$\begin{array}{ll} (H_1) & X \text{ is a UMD space}, \\ (H_2) & 0 \in \rho(A), \\ (H_3) & -A \in \mathrm{BIP}(X, \theta_A) \text{ for some } \theta_A \in (0, \pi/2), \\ (H_4) & -A \in \mathrm{Sect}(0). \end{array}$$

Note that assumption (H_4) means that $-A$ is a sectorial operator of any angle $\theta \in (0, \pi)$. This assumption is satisfied, for instance, by elliptic differential operators of second order. Now, we give some consequences on our hypotheses.

3.4 Consequences

1. Note that A_0 satisfies all the previous hypotheses with $X = L^q(\omega)$, with $q \in (1, +\infty)$. From [29], Proposition 3, p. 207, X satisfies (H_1) and from [11], Theorem 9.15, p. 241 and Lemma 9.17, p. 242, A_0 satisfies (H_2). Moreover, from [28], Theorem C, p. 166-167, (H_3) is satisfied for every $\theta_A \in (0, \pi)$, thus (H_4) is also satisfied.

2. In the scalar case, to solve each equation of (EQ), it is necessary to introduce the square roots $\pm\sqrt{-A}$ of the characteristic equations

$$x^4 + 2Ax^2 + A^2 = 0,$$

this is why, in our operational case, we consider,

$$M \ := \ -\sqrt{-A}. \tag{5}$$

From (H_3), $-A$ is a sectorial operator, thus the existence of M is ensured, see for instance [14], e), p. 25.

3. From (H_3), we have $-A \in \mathrm{BIP}(X, \theta_A)$, then, from [14], Proposition 3.2.1, e), p. 71, we deduce

$$-M \in \mathrm{BIP}(X, \theta_A/2).$$

Since $0 < \theta_A/2 < \pi/2$, we deduce that M generate a bounded analytic semigroup $(e^{xM})_{x\geqslant 0}$, see [27], Theorem 2, p. 437. Furthermore, due to [27], Theorem 4, p. 441, for $n \in \mathbb{N} \setminus \{0\}$, we get $-nM \in \mathrm{BIP}(X, \theta_A/4 + \varepsilon)$, for any $\varepsilon \in (0, \pi/2 - \theta_A/4)$. Then, due to [27], Theorem 2, p. 437, nM generate a bounded analytic semigroup $(e^{nxM})_{x\geqslant 0}$. The last results use also the works of [7] and [8].

3.5 The main results

To solve our transmission problem (P), we introduce two auxiliary problems:

$$(P_-)\begin{cases} u_-^{(4)}(x) + 2Au_-''(x) + A^2u_-(x) = f_-(x), & \text{for a.e. } x \in (a,\gamma) \\ u_-(a) \;=\; \varphi_1^-, \quad u_-(\gamma) \;=\; \psi_1 \\ u_-'(a) \;=\; \varphi_2^-, \quad u_-'(\gamma) \;=\; \psi_2, \end{cases}$$

and

$$(P_+)\begin{cases} u_+^{(4)}(x) + 2Au_+''(x) + A^2u_+(x) = f_+(x), & \text{for a.e. } x \in (\gamma,b) \\ u_+(\gamma) \;=\; \psi_1, \quad u_+(b) \;=\; \varphi_1^+ \\ u_+'(\gamma) \;=\; \psi_2, \quad u_+'(b) \;=\; \varphi_2^+. \end{cases}$$

Remark 3.8. Recall that a classical solution of $(P_\pm)$, in $L^p(J;X)$, with $J=(a,\gamma)$ or (γ,b), is a solution to $(P_\pm)$ such that

$$u_\pm \in W^{4,p}(J;X) \cap L^p(J;D(A^2)), \quad u'' \in L^p(J;D(A)).$$

We say that u is a classical solution of (P) if and only if there exist $\psi_1,\psi_2 \in X$ such that

$$\begin{array}{ll} (i) & u_- \text{ is a classical solution of } (P_-), \\ (ii) & u_+ \text{ is a classical solution of } (P_+), \\ (iii) & u_- \text{ and } u_+ \text{ satisfy } (TC2). \end{array}$$

Note that by construction, if there exist a classical solution u_- of (P_-) and u_+ of (P_+), then u_- and u_+ satisfy $(TC1)$.

Our goal is to prove that there exists a unique couple (ψ_1,ψ_2) which satisfies (i), (ii) and (iii). This will lead us to obtain our main result.

Theorem 3.9. Let $f_- \in L^p(a,\gamma;X)$ and $f_+ \in L^p(\gamma,b;X)$. Assume that (H_1), (H_2) and (H_3) be satisfied. Then, there exists a unique classical solution u of the transmission problem (P) if and only if

$$\varphi_1^+,\varphi_1^- \in (D(A),X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_2^+,\varphi_2^- \in (D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{6}$$

Remark 3.10.

1. In the proof of Theorem 3.9, we use operator M and interpolation spaces $(D(M),X)_{3-j+\frac{1}{p},p}$, $j=0,1,2,3$. But from the reiteration Theorem, we get

$$\begin{cases} (D(M),X)_{3+\frac{1}{p},p} \;=\; (D(A),X)_{1+\frac{1}{2p},p}, & (D(M),X)_{2+\frac{1}{p},p} \;=\; (D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p} \\ (D(M),X)_{1+\frac{1}{p},p} \;=\; (D(A),X)_{\frac{1}{2p},p}, & (D(M),X)_{\frac{1}{p},p} \;=\; (D(A),X)_{\frac{1}{2}+\frac{1}{2p},p}. \end{cases} \tag{7}$$

2. We can generalize the previous Theorem by considering a transmission problem between N juxtaposed habitats, with $N \in \mathbb{N}\setminus\{0\}$. For instance, with $N=3$, it suffices to use Theorem 3.9 on the two first habitats and then apply it on the transmission problem between the second and third habitat to solve the problem. By recurrence, we obtain the result for N habitats.

As consequence of Theorem 3.9, we deduce the following result for problem (P_{pde}). Consider the case $A=A_0$ (other cases can be treated).

Corollary 3.11. Assume that ω is a bounded open set of $\mathbb{R}^{n-1}$ where $n \geq 2$ with C^2-boundary. Let $g_+ \in L^p(\Omega_+)$ and $g_- \in L^p(\Omega_-)$ with $p \in (1, +\infty)$ and $p > n$; let $k_+, k_- \in \mathbb{R}_+ \setminus \{0\}$. Then, there exists a unique solution u of (P_{pde}), such that

$$u_- \in W^{4,p}(\Omega_-), \quad u_+ \in W^{4,p}(\Omega_+),$$

if and only if

$$\varphi_1^{\pm}, \varphi_2^{\pm} \in W^{2,p}(\omega) \cap W_0^{1,p}(\omega), \quad \Delta\varphi_1^{\pm}, \in W^{2-\frac{1}{p},p}(\omega) \cap W_0^{1,p}(\omega) \text{ and } \Delta\varphi_2^{\pm} \in W^{1-\frac{1}{p},p}(\omega) \cap W_0^{1,p}(\omega).$$

Proof. The proof is quite similar to the one of Corollary 3.6 in [20], see also Corollary 2.7 in [19]. □

Taking into account the result of Theorem 3.9, we can also obtain anisotropic results by considering $f_- \in L^p(a, \gamma; L^q(\omega))$ and $f_+ \in L^p(\gamma, b; L^q(\omega))$ with $p, q \in (1, +\infty)$.

4 Preliminary results

Throughout this article, we set

$$c = \gamma - a > 0 \quad \text{and} \quad d = b - \gamma > 0.$$

From Remark 3.8, to solve problem (P), we first have to study problems (P_-) and (P_+). To this end, we need the following invertibility result obtained in [31].

Lemma 4.1 ([31])**.** The operators U_+, U_-, V_+, $V_- \in \mathcal{L}(X)$ defined by

$$\left\{\begin{array}{lllllll} U_- &:=& I - e^{2cM} + 2cMe^{cM}, & U_+ &:=& I - e^{2dM} + 2dMe^{dM} \\ V_- &:=& I - e^{2cM} - 2cMe^{cM}, & V_+ &:=& I - e^{2dM} - 2dMe^{dM}, \end{array}\right. \tag{8}$$

are invertible with bounded inverse.

All these exponentials are analytic semigroups and they are well defined due to statement 3 of Section 3.4. For a detailed proof, see [31], Proposition 4.5.

4.1 Problem $(\mathbf{P}_-)$

Proposition 4.2. Let $f_- \in L^p(a, \gamma; X)$. Assume that (H_1), (H_2) and (H_3) hold. Then there exists a unique classical solution u_- of problem (P_-) if and only if

$$\varphi_1^-, \psi_1 \in (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_2^-, \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{9}$$

Moreover

$$\begin{aligned} u_-(x) \;=\; & \left(e^{(x-a)M} - e^{(\gamma-x)M}\right)\alpha_1^- + \left((x-a)e^{(x-a)M} - (\gamma-x)e^{(\gamma-x)M}\right)\alpha_2^- \\ & + \left(e^{(x-a)M} + e^{(\gamma-x)M}\right)\alpha_3^- + \left((x-a)e^{(x-a)M} + (\gamma-x)e^{(\gamma-x)M}\right)\alpha_4^- + F_-(x), \end{aligned} \tag{10}$$

where

$$\left\{\begin{array}{lll} \alpha_1^- &:=& -\dfrac{1}{2}U_-^{-1}\left(\left(I + (I + cM)\, e^{cM}\right)\psi_1 - ce^{cM}\psi_2\right) + \tilde{\varphi}_1^- \\ \alpha_2^- &:=& \dfrac{1}{2}U_-^{-1}\left(\left(I + e^{cM}\right)M\psi_1 + \left(I - e^{cM}\right)\psi_2\right) + \tilde{\varphi}_2^- \\ \alpha_3^- &:=& \dfrac{1}{2}V_-^{-1}\left(\left(I - (I + cM)\, e^{cM}\right)\psi_1 + ce^{cM}\psi_2\right) + \tilde{\varphi}_3^- \\ \alpha_4^- &:=& -\dfrac{1}{2}V_-^{-1}\left(\left(I - e^{cM}\right)M\psi_1 + \left(I + e^{cM}\right)\psi_2\right) + \tilde{\varphi}_4^-, \end{array}\right. \tag{11}$$

with

$$
\left\{
\begin{array}{rcl}
\tilde{\varphi_1}^- & := & \dfrac{1}{2}U_-^{-1}\left(\varphi_1^- + e^{cM}\left(\varphi_1^- + c\left(M\varphi_1^- + \varphi_2^- - F_-'(a) - F_-'(\gamma)\right)\right)\right) \\[2ex]
\tilde{\varphi_2}^- & := & -\dfrac{1}{2}U_-^{-1}\left(M\varphi_1^- - \varphi_2^- + F_-'(a) + F_-'(\gamma)\right) \\
 & & -\dfrac{1}{2}U_-^{-1}e^{cM}\left(M\varphi_1^- + \varphi_2^- - F_-'(a) - F_-'(\gamma)\right) \\[2ex]
\tilde{\varphi_3}^- & := & \dfrac{1}{2}V_-^{-1}\left(\varphi_1^- - e^{cM}\left(\varphi_1^- + c\left(M\varphi_1^- + \varphi_2^- - F_-'(a) + F_-'(\gamma)\right)\right)\right) \\[2ex]
\tilde{\varphi_4}^- & := & -\dfrac{1}{2}V_-^{-1}\left(M\varphi_1^- - \varphi_2^- + F_-'(a) - F_-'(\gamma)\right) \\
 & & +\dfrac{1}{2}V_-^{-1}e^{cM}\left(M\varphi_1^- + \varphi_2^- - F_-'(a) + F_-'(\gamma)\right),
\end{array}
\right. \tag{12}
$$

and F_- is the unique classical solution of problem

$$
\left\{
\begin{array}{l}
u_-^{(4)}(x) + 2Au_-''(x) + A^2u_-(x) = f_-(x), \quad \text{a.e. } x \in (a,\gamma) \\
u_-(a) = u_-(\gamma) = 0 \\
u_-''(a) = u_-''(\gamma) = 0.
\end{array}
\right. \tag{13}
$$

Proof. Due to [31], Theorem 2.8, statement 2., there exists a unique classical solution u_- of problem (P_-) if and only if (9) holds.

Moreover, adapting the representation formula of u given by (31) in [31], where u, f, b, $F_{0,f}$, φ_1, φ_2, φ_3 and φ_4 are respectively replaced by u_-, f_-, γ, F_-, φ_1^-, ψ_1, φ_2^- and ψ_2, we obtain that the representation formula of u_- is given by (10), (11) and (12). $\square$

Remark 4.3. In the previous proposition, since (9), (11) and (12) hold, we have

$$
\alpha_1^-, \alpha_3^- \in D(M^3) \quad \text{and} \quad \alpha_2^-, \alpha_4^- \in D(M^2).
$$

Moreover, since F_- is a classical solution of (13), due to Lemma 3.7, for $j = 0, 1, 2, 3$ and $s = a$ or γ

$$
F_-^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{p},p}.
$$

4.2 Problem $(\mathbf{P}_+)$

Proposition 4.4. Let $f_+ \in L^p(\gamma, b; X)$. Assume that (H_1), (H_2) and (H_3) hold. Then, there exists a unique classical solution u_+ of problem (P_+) if and only if

$$
\varphi_1^+, \psi_1 \in (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_2^+, \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{14}
$$

Moreover

$$
\begin{array}{rcl}
u_+(x) & = & \left(e^{(x-\gamma)M} - e^{(b-x)M}\right)\alpha_1^+ + \left((x-\gamma)e^{(x-\gamma)M} - (b-x)e^{(b-x)M}\right)\alpha_2^+ \\
 & & + \left(e^{(x-\gamma)M} + e^{(b-x)M}\right)\alpha_3^+ + \left((x-\gamma)e^{(x-\gamma)M} + (b-x)e^{(b-x)M}\right)\alpha_4^+ \\
 & & + F_+(x),
\end{array} \tag{15}
$$

where

$$\begin{cases} \alpha_1^+ &:=& \dfrac{1}{2}U_+^{-1}\left(\left(I+(I+dM)\,e^{dM}\right)\psi_1 + de^{dM}\psi_2\right) + \tilde{\varphi}_1^+ \\ \alpha_2^+ &:=& -\dfrac{1}{2}U_+^{-1}\left(\left(I+e^{dM}\right)M\psi_1 - \left(I-e^{dM}\right)\psi_2\right) + \tilde{\varphi}_2^+ \\ \alpha_3^+ &:=& \dfrac{1}{2}V_+^{-1}\left(\left(I-(I+dM)\,e^{dM}\right)\psi_1 - de^{dM}\psi_2\right) + \tilde{\varphi}_3^+ \\ \alpha_4^+ &:=& -\dfrac{1}{2}V_+^{-1}\left(\left(I-e^{dM}\right)M\psi_1 - \left(I+e^{dM}\right)\psi_2\right) + \tilde{\varphi}_4^+, \end{cases} \tag{16}$$

with

$$\begin{cases} \tilde{\varphi_1}^+ &:=& -\dfrac{1}{2}U_+^{-1}\left(\varphi_1^+ + e^{dM}\left(\varphi_1^+ + d\left(M\varphi_1^+ - \varphi_2^+ + F_+'(\gamma) + F_+'(b)\right)\right)\right) \\ \tilde{\varphi_2}^+ &:=& \dfrac{1}{2}U_+^{-1}\left(M\varphi_1^+ + \varphi_2^+ - F_+'(\gamma) - F_+'(b)\right) \\ && +\dfrac{1}{2}U_+^{-1}e^{dM}\left(M\varphi_1^+ - \varphi_2^+ + F_+'(\gamma) + F_+'(b)\right) \\ \tilde{\varphi_3}^+ &:=& \dfrac{1}{2}V_+^{-1}\left(\varphi_1^+ - e^{dM}\left(\varphi_1^+ + d\left(M\varphi_1^+ - \varphi_2^+ - F_+'(\gamma) + F_+'(b)\right)\right)\right) \\ \tilde{\varphi_4}^+ &:=& -\dfrac{1}{2}V_+^{-1}\left(M\varphi_1^+ + \varphi_2^+ + F_+'(\gamma) - F_+'(b)\right) \\ && +\dfrac{1}{2}V_+^{-1}e^{dM}\left(M\varphi_1^+ - \varphi_2^+ - F_+'(\gamma) + F_+'(b)\right), \end{cases} \tag{17}$$

and F_+ is the unique classical solution of problem

$$\begin{cases} u_+^{(4)}(x) + 2Au_+''(x) + A^2u_+(x) = f_+(x), \quad \text{a.e. } x \in (\gamma, b) \\ u_+(\gamma) = u_+(b) = u_+''(\gamma) = u_+''(b) = 0. \end{cases} \tag{18}$$

Proof. Due to [31], Theorem 2.8, statement 2., there exists a unique classical solution u_+ of problem (P_+) if and only if (14) holds. Moreover, adapting the representation formula of u given by (31) in [31], where u, f, a, $F_{0,f}$, φ_1, φ_2, φ_3 and φ_4 are respectively replaced by u_+, f_+, γ, F_+, ψ_1, φ_1^+, ψ_2 and φ_2^+, we obtain that the representation formula of u_+ is given by (15), (16) and (17). □

Remark 4.5. In the previous proposition, since (14), (16) and (17) hold, we have

$$\alpha_1^+, \alpha_3^+ \in D(M^3) \quad \text{and} \quad \alpha_2^+, \alpha_4^+ \in D(M^2).$$

Moreover, since F_+ is a classical solution of (18), due to Lemma 3.7, for $j = 0, 1, 2, 3$ and $s = \gamma$ or b

$$F_+^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{p},p}.$$

4.3 The transmission system

In this section we give the proof of Theorem 4.6 stated below. This theorem ensures the equivalence between the resolution of problem (P) and the resolution of the following system

$$\begin{cases} \left(P_1^+ + P_1^-\right)M\psi_1 - \left(P_2^+ - P_2^-\right)\psi_2 &=& S_1 \\ \left(P_2^+ - P_2^-\right)M\psi_1 - \left(P_3^+ + P_3^-\right)\psi_2 &=& S_2. \end{cases} \tag{19}$$

The coefficients of the previous system are given by

$$\begin{cases} P_1^+ &=& k_+\left(U_+^{-1}\left(I+e^{dM}\right)^2+V_+^{-1}\left(I-e^{dM}\right)^2\right) \\ P_2^+ &=& k_+\left(U_+^{-1}+V_+^{-1}\right)\left(I-e^{2dM}\right) \\ P_3^+ &=& k_+\left(U_+^{-1}\left(I-e^{dM}\right)^2+V_+^{-1}\left(I+e^{dM}\right)^2\right) \end{cases} \tag{20}$$

and

$$\begin{cases} P_1^- &=& k_-\left(U_-^{-1}\left(I+e^{cM}\right)^2+V_-^{-1}\left(I-e^{cM}\right)^2\right) \\ P_2^- &=& k_-\left(U_-^{-1}+V_-^{-1}\right)\left(I-e^{2cM}\right) \\ P_3^- &=& k_-\left(U_-^{-1}\left(I-e^{cM}\right)^2+V_-^{-1}\left(I+e^{cM}\right)^2\right). \end{cases} \tag{21}$$

The seconds members are given by

$$S_1 = 2k_+\left(\left(\tilde\varphi_2^+ + \tilde\varphi_4^+\right)+e^{dM}\left(\tilde\varphi_2^+ - \tilde\varphi_4^+\right)\right) - 2k_-\left(\left(\tilde\varphi_2^- - \tilde\varphi_4^-\right)+e^{cM}\left(\tilde\varphi_2^- + \tilde\varphi_4^-\right)\right) - M^{-2}\check S \tag{22}$$

where

$$\check S = -k_+F_+^{(3)}(\gamma)+k_+M^2F_+'(\gamma)+k_-F_-^{(3)}(\gamma)-k_-M^2F_-'(\gamma) \tag{23}$$

and

$$S_2 = 2k_+\left(\left(\tilde\varphi_2^+ + \tilde\varphi_4^+\right)-e^{dM}\left(\tilde\varphi_2^+ - \tilde\varphi_4^+\right)\right) + 2k_-\left(\left(\tilde\varphi_2^- - \tilde\varphi_4^-\right)-e^{cM}\left(\tilde\varphi_2^- + \tilde\varphi_4^-\right)\right). \tag{24}$$

Theorem 4.6. Let $f_- \in L^p(a,\gamma;X)$ and $f_+ \in L^p(\gamma,b;X)$. Assume that (H_1), (H_2) and (H_3) hold. Then, problem (P) has a unique classical solution if and only if the data φ_1^+, φ_1^-, φ_2^+, φ_2^- satisfy (6) and system (19) has a unique solution (ψ_1,ψ_2) such that

$$(\psi_1,\psi_2) \in (D(A),X)_{1+\frac{1}{2p},p} \times (D(A),X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{25}$$

Proof. Assume that problem (P) has a unique classical solution u. We set

$$\psi_1 = u_-(\gamma) = u_+(\gamma) \quad \text{and} \quad \psi_2 = u_-'(\gamma) = u_+'(\gamma).$$

We get that u_- (respectively u_+) is the classical solution of (P_-) (respectively (P_+)). Then, applying Proposition 4.2 (respectively Proposition 4.4), we obtain (6). Moreover, from (7), we have

$$\psi_1 \in (D(M),X)_{3+\frac{1}{p},p} \quad \text{and} \quad \psi_2 \in (D(M),X)_{2+\frac{1}{p},p}.$$

It remains to prove that (ψ_1,ψ_2) is solution of system (19). To this end, since u satisfies the transmission conditions $(TC2)$. Then, we obtain the following system

$$\begin{cases} k_-u_-^{(3)}(\gamma) &+& k_-Au_-'(\gamma) &=& k_+u_+^{(3)}(\gamma) &+& k_+Au_+'(\gamma) \\ k_-u_-''(\gamma) &+& k_-Au_-(\gamma) &=& k_+u_+''(\gamma) &+& k_+Au_+(\gamma). \end{cases}$$

Hence

$$\begin{cases} k_+\left(u_+^{(3)}(\gamma)-M^2u_+'(\gamma)\right) &-& k_-\left(u_-^{(3)}(\gamma)-M^2u_-'(\gamma)\right) &=& 0 \\ k_+\left(u_+''(\gamma)-M^2u_+(\gamma)\right) &-& k_-\left(u_-''(\gamma)-M^2u_-(\gamma)\right) &=& 0. \end{cases}$$

Moreover, for all $x \in (a, \gamma)$, we have

$$\begin{aligned}
u_-(x) &= \left(e^{(x-a)M} - e^{(\gamma-x)M}\right)\alpha_1^- + \left((x-a)e^{(x-a)M} - (\gamma-x)e^{(\gamma-x)M}\right)\alpha_2^- \\
&\quad + \left(e^{(x-a)M} + e^{(\gamma-x)M}\right)\alpha_3^- + \left((x-a)e^{(x-a)M} + (\gamma-x)e^{(\gamma-x)M}\right)\alpha_4^- \\
&\quad + F_-(x) \\
u_-'(x) &= M\left(e^{(x-a)M} + e^{(\gamma-x)M}\right)\alpha_1^- + M\left(e^{(x-a)M} - e^{(\gamma-x)M}\right)\alpha_3^- \\
&\quad + \left((I+(x-a)M)e^{(x-a)M} + (I+(\gamma-x)M)e^{(\gamma-x)M}\right)\alpha_2^- \\
&\quad + \left((I+(x-a)M)e^{(x-a)M} - (I+(\gamma-x)M)e^{(\gamma-x)M}\right)\alpha_4^- + F_-'(x) \\
u_-''(x) &= M^2\left(e^{(x-a)M} - e^{(\gamma-x)M}\right)\alpha_1^- + M^2\left(e^{(x-a)M} + e^{(\gamma-x)M}\right)\alpha_3^- + F_-''(x) \\
&\quad + \left((2M+(x-a)M^2)e^{(x-a)M} - (2M+(\gamma-x)M^2)e^{(\gamma-x)M}\right)\alpha_2^- \\
&\quad + \left((2M+(x-a)M^2)e^{(x-a)M} + (2M+(\gamma-x)M^2)e^{(\gamma-x)M}\right)\alpha_4^-, \\
u_-^{(3)}(x) &= M^3\left(e^{(x-a)M} + e^{(\gamma-x)M}\right)\alpha_1^- + M^3\left(e^{(x-a)M} - e^{(\gamma-x)M}\right)\alpha_3^- + F_-^{(3)}(x) \\
&\quad + \left((3M^2+(x-a)M^3)e^{(x-a)M} + (3M^2+(\gamma-x)M^3)e^{(\gamma-x)M}\right)\alpha_2^- \\
&\quad + \left((3M^2+(x-a)M^3)e^{(x-a)M} - (3M^2+(\gamma-x)M^3)e^{(\gamma-x)M}\right)\alpha_4^-.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
k_-\left(u_-^{(3)}(\gamma) - M^2 u_-'(\gamma)\right) &= k_-\left(2M^2\left(I+e^{cM}\right)\alpha_2^- - 2M^2\left(I-e^{cM}\right)\alpha_4^-\right) \\
&\quad + k_- F_-^{(3)}(\gamma) - k_- M^2 F_-'(\gamma).
\end{aligned} \tag{26}$$

Moreover, from (13), we have

$$k_-\left(u_-''(\gamma) - M^2 u_-(\gamma)\right) = -k_-\left(2M\left(I-e^{cM}\right)\alpha_2^- - 2M\left(I+e^{cM}\right)\alpha_4^-\right). \tag{27}$$

Note that, from Remark 4.3, all terms of the previous equality are well defined.

In the same way, for all $x \in (\gamma, b)$, we have

$$\begin{aligned}
u_+(x) &= \left(e^{(x-\gamma)M} - e^{(b-x)M}\right)\alpha_1^+ + \left((x-\gamma)e^{(x-\gamma)M} - (b-x)e^{(b-x)M}\right)\alpha_2^+ \\
&\quad + \left(e^{(x-\gamma)M} + e^{(b-x)M}\right)\alpha_3^+ + \left((x-\gamma)e^{(x-\gamma)M} + (b-x)e^{(b-x)M}\right)\alpha_4^+ \\
&\quad + F_+(x), \\
u_+'(x) &= M\left(e^{(x-\gamma)M} + e^{(b-x)M}\right)\alpha_1^+ + M\left(e^{(x-\gamma)M} - e^{(b-x)M}\right)\alpha_3^+ + F_+'(x) \\
&\quad + \left((I+(x-\gamma)M)e^{(x-\gamma)M} + (I+(b-x)M)e^{(b-x)M}\right)\alpha_2^+ \\
&\quad + \left((I+(x-\gamma)M)e^{(x-\gamma)M} - (I+(b-x)M)e^{(b-x)M}\right)\alpha_4^+,
\end{aligned}$$

and

$$
\begin{aligned}
u_+''(x) &= M^2\left(e^{(x-\gamma)M}-e^{(b-x)M}\right)\alpha_1^+ + M^2\left(e^{(x-\gamma)M}+e^{(b-x)M}\right)\alpha_3^+ \\
&\quad+\left((2M+(x-\gamma)M^2)e^{(x-\gamma)M}-(2M+(b-x)M^2)e^{(b-x)M}\right)\alpha_2^+ \\
&\quad+\left((2M+(x-\gamma)M^2)e^{(x-\gamma)M}+(2M+(b-x)M^2)e^{(b-x)M}\right)\alpha_4^+ \\
&\quad+F_+''(x), \\
u_+^{(3)}(x) &= M^3\left(e^{(x-\gamma)M}+e^{(b-x)M}\right)\alpha_1^+ + M^3\left(e^{(x-\gamma)M}-e^{(b-x)M}\right)\alpha_3^+ \\
&\quad+\left((3M^2+(x-\gamma)M^3)e^{(x-\gamma)M}+(3M^2+(b-x)M^3)e^{(b-x)M}\right)\alpha_2^+ \\
&\quad+\left((3M^2+(x-\gamma)M^3)e^{(x-\gamma)M}-(3M^2+(b-x)M^3)e^{(b-x)M}\right)\alpha_4^+ \\
&\quad+F_+^{(3)}(x).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
k_+\left(u_+^{(3)}(\gamma)-M^2u_+'(\gamma)\right) &= k_+\left(2M^2\left(I+e^{dM}\right)\alpha_2^+ + 2M^2\left(I-e^{dM}\right)\alpha_4^+\right) \\
&\quad+k_+F_+^{(3)}(\gamma)-k_+M^2F_+'(\gamma).
\end{aligned}
\tag{28}
$$

Moreover, from (18), we have

$$
k_+\left(u_+''(\gamma)-M^2u_+(\gamma)\right) = k_+\left(2M\left(I-e^{dM}\right)\alpha_2^+ + 2M\left(I+e^{dM}\right)\alpha_4^+\right). \tag{29}
$$

As previously, from Remark 4.5, all terms of the previous equality are justified. Then, from (23), (26), (27), (28) and (29), we deduce that system (19) is equivalent to

$$
\left\{
\begin{aligned}
& k_+\left(2M^2\left(I+e^{dM}\right)\alpha_2^+ + 2M^2\left(I-e^{dM}\right)\alpha_4^+\right) \\
& -k_-\left(2M^2\left(I+e^{cM}\right)\alpha_2^- - 2M^2\left(I-e^{cM}\right)\alpha_4^-\right) &&= \check{S} \\
& k_+\left(2M\left(I-e^{dM}\right)\alpha_2^+ + 2M\left(I+e^{dM}\right)\alpha_4^+\right) \\
& +k_-\left(2M\left(I-e^{cM}\right)\alpha_2^- - 2M\left(I+e^{cM}\right)\alpha_4^-\right) &&= 0.
\end{aligned}
\right.
$$

Thus, we obtain the following system

$$
\left\{
\begin{aligned}
& k_+\left(\left(I+e^{dM}\right)\alpha_2^+ + \left(I-e^{dM}\right)\alpha_4^+\right) \\
& -k_-\left(\left(I+e^{cM}\right)\alpha_2^- - \left(I-e^{cM}\right)\alpha_4^-\right) &&= \frac{1}{2}M^{-2}\check{S} \\
& k_+\left(\left(I-e^{dM}\right)\alpha_2^+ + \left(I+e^{dM}\right)\alpha_4^+\right) \\
& +k_-\left(\left(I-e^{cM}\right)\alpha_2^- - \left(I+e^{cM}\right)\alpha_4^-\right) &&= 0.
\end{aligned}
\right.
\tag{30}
$$

Now, we regroup all source terms provided from (11), (12), (16) and (17) in S_1 and S_2, defined by (22), (23) and (24). Then, we deduce that system (30) writes

$$\left\{\begin{array}{rcl}
-\dfrac{k_+}{2}U_+^{-1}\left(I+e^{dM}\right)\left(\left(I+e^{dM}\right)M\psi_1-\left(I-e^{dM}\right)\psi_2\right) & & \\
-\dfrac{k_+}{2}V_+^{-1}\left(I-e^{dM}\right)\left(\left(I-e^{dM}\right)M\psi_1-\left(I+e^{dM}\right)\psi_2\right) & & \\
-\dfrac{k_-}{2}U_-^{-1}\left(I+e^{cM}\right)\left(\left(I+e^{cM}\right)M\psi_1+\left(I-e^{cM}\right)\psi_2\right) & & \\
-\dfrac{k_-}{2}V_-^{-1}\left(I-e^{cM}\right)\left(\left(I-e^{cM}\right)M\psi_1+\left(I+e^{cM}\right)\psi_2\right) & = & -\dfrac{S_1}{2} \\
-\dfrac{k_+}{2}U_+^{-1}\left(I-e^{dM}\right)\left(\left(I+e^{dM}\right)M\psi_1-\left(I-e^{dM}\right)\psi_2\right) & & \\
-\dfrac{k_+}{2}V_+^{-1}\left(I+e^{dM}\right)\left(\left(I-e^{dM}\right)M\psi_1-\left(I+e^{dM}\right)\psi_2\right) & & \\
+\dfrac{k_-}{2}U_-^{-1}\left(I-e^{cM}\right)\left(\left(I+e^{cM}\right)M\psi_1+\left(I-e^{cM}\right)\psi_2\right) & & \\
+\dfrac{k_-}{2}V_-^{-1}\left(I+e^{cM}\right)\left(\left(I-e^{cM}\right)M\psi_1+\left(I+e^{cM}\right)\psi_2\right) & = & -\dfrac{S_2}{2}.
\end{array}\right.$$

Finally, we obtain

$$\left\{\begin{array}{rcl}
k_+\left(U_+^{-1}\left(I+e^{dM}\right)^2+V_+^{-1}\left(I-e^{dM}\right)^2\right)M\psi_1 & & \\
+k_-\left(U_-^{-1}\left(I+e^{cM}\right)^2+V_-^{-1}\left(I-e^{cM}\right)^2\right)M\psi_1 & & \\
-k_+\left(U_+^{-1}+V_+^{-1}\right)\left(I-e^{2dM}\right)\psi_2 & & \\
+k_-\left(U_-^{-1}+V_-^{-1}\right)\left(I-e^{2cM}\right)\psi_2 & = & S_1 \\
k_+\left(U_+^{-1}+V_+^{-1}\right)\left(I-e^{2dM}\right)M\psi_1 & & \\
-k_-\left(U_-^{-1}+V_-^{-1}\right)\left(I-e^{2cM}\right)M\psi_1 & & \\
-k_+\left(U_+^{-1}\left(I-e^{dM}\right)^2+V_+^{-1}\left(I+e^{dM}\right)^2\right)\psi_2 & & \\
-k_-\left(U_-^{-1}\left(I-e^{cM}\right)^2+V_-^{-1}\left(I+e^{cM}\right)^2\right)\psi_2 & = & S_2.
\end{array}\right. \tag{31}$$

Then, using (20) and (21) system (31) become system (19). So, (ψ_1, ψ_2) is solution of system (19).

Conversely, if (6) holds and system (19) has a unique solution (ψ_1, ψ_2) such that

$$\psi_1 \in (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p},$$

Then, considering u_- (respectively u_+) the unique classical solution of problem (P_-) (respectively problem (P_+)), we obtain that

$$u = \left\{\begin{array}{ll} u_- & \text{in } \Omega_- \\ u_+ & \text{in } \Omega_+, \end{array}\right.$$

is the unique classical solution of problem (P). $\square$

4.4 Functional calculus

To prove Theorem 3.9, it remains, from Theorem 4.6, to solve system (19). To this end, we have to show that the determinant operator of system (19) is an invertible operator using functional calculus.

We first recall some classical notations. Let $\theta \in (0, \pi)$, we denote by $H(S_\theta)$ the space of holomorphic functions on S_θ (defined by (3)). Moreover, we consider the following subspace of $H(S_\theta)$:

$$\mathcal{E}_\infty(S_\theta) := \left\{ f \in H(S_\theta) : f = O(|z|^{-s}) \; (|z| \to +\infty) \text{ for some } s > 0 \right\}.$$

Thus, $\mathcal{E}_\infty(S_\theta)$ is the space of polynomially decreasing holomorphic functions at $+\infty$.

Let T be an invertible sectorial operator of angle $\theta_T \in (0, \pi)$ and let $f \in \mathcal{E}_\infty(S_\theta)$, with $\theta \in (\theta_T, \pi)$, then, by functional calculus, we can define $f(T) \in \mathcal{L}(X)$, see [14], p. 45.

We recall a useful invertibility result from [19], Lemma 5.3.

Lemma 4.7 ([19])**.** Let P be an invertible sectorial operator in X with angle θ, for all $\theta \in (0, \pi)$. Let $G \in H(S_\theta)$, for some $\theta \in (0, \pi)$, such that

(i) $1 - G \in \mathcal{E}_\infty(S_\theta)$,

(ii) $G(x) \neq 0$ for any $x \in \mathbb{R}_+ \setminus \{0\}$.

Then, $G(P) \in \mathcal{L}(X)$, is invertible with bounded inverse.

Now, to inverse the determinant of system (19), we introduce some holomorphic functions which will play the role of G in the previous lemma. Then, we study them on the positive real axis.

Let $\delta > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}_-$. We set

$$\begin{cases} u_\delta(z) &=& 1 - e^{-2\delta\sqrt{z}} - 2\delta\sqrt{z}e^{-\delta\sqrt{z}} \\ v_\delta(z) &=& 1 - e^{-2\delta\sqrt{z}} + 2\delta\sqrt{z}e^{-\delta\sqrt{z}}. \end{cases}$$

Then, we have

$$U_+ = u_d(-A), \quad U_- = u_c(-A), \quad V_+ = v_d(-A) \quad \text{and} \quad V_- = v_c(-A).$$

We set

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant 0\}.$$

Moreover, from [31], Lemma 4.4, we have $u_\delta(z) \neq 0$ and $v_\delta(z) \neq 0$ for $z \in \mathbb{C}_+ \setminus \{0\}$, thus we note

$$\begin{cases} f_{\delta,1}(z) &=& u_\delta^{-1}(z)\left(1 + e^{-\delta\sqrt{z}}\right)^2 + v_\delta^{-1}(z)\left(1 - e^{-\delta\sqrt{z}}\right)^2 \\ f_{\delta,2}(z) &=& \left(u_\delta^{-1}(z) + v_\delta^{-1}(z)\right)\left(1 - e^{-2\delta\sqrt{z}}\right) \\ f_{\delta,3}(z) &=& u_\delta^{-1}(z)\left(1 - e^{-\delta\sqrt{z}}\right)^2 + v_\delta^{-1}(z)\left(1 + e^{-\delta\sqrt{z}}\right)^2 \\ g_\delta(z) &=& 16\, u_\delta^{-1}(z) v_\delta^{-1}(z) e^{-2\delta\sqrt{z}}. \end{cases}$$

Thus, we obtain that

$$\begin{cases} P_1^+ = k_+ f_{d,1}(-A), \quad P_2^+ = k_+ f_{d,2}(-A), \quad P_3^+ = k_+ f_{d,3}(-A) \\ P_1^- = k_- f_{c,1}(-A), \quad P_2^- = k_- f_{c,2}(-A), \quad P_3^- = k_- f_{c,3}(-A). \end{cases}$$

Moreover, for $z \in \mathbb{C}_+ \setminus \{0\}$, we define

$$f(z) := k_+^2 g_d(z) + k_-^2 g_c(z) + k_+ k_- \left(f_{d,1}(z) f_{c,3}(z) + f_{c,1}(z) f_{d,3}(z) + 2 f_{d,2}(z) f_{c,2}(z) \right).$$

Note that $f \in H(S_\theta)$, for all $\theta \in (0, \pi)$. It follows that

$$f(-A) = 16 k_+^2 U_+^{-1} V_+^{-1} e^{2dM} + 16 k_-^2 U_-^{-1} V_-^{-1} e^{2cM} + P_1^+ P_3^- + P_1^- P_3^+ + 2 P_2^+ P_2^-. \tag{32}$$

Lemma 4.8. For all $x \in \mathbb{R}_+ \setminus \{0\}$, then $f(x)$ does not vanish.

Proof. Let $\delta > 0$, from Lemma 4.4, in [31], for all $z \in \mathbb{C}_+ \setminus \{0\}$, we have

$$0 < |u_\delta(z)| \quad \text{and} \quad 0 < |v_\delta(z)|.$$

Thus, for all $x > 0$, one has $u_\delta(x)$, $v_\delta(x) > 0$. Moreover, for all $x > 0$, we deduce that

$$f_{\delta,1}(x), f_{\delta,2}(x), f_{\delta,3}(x), g_\delta(x) > 0$$

and since $k_+ k_- > 0$, it follows that

$$f(x) = k_+^2 g_d(x) + k_-^2 g_c(x) + k_+ k_- \left(f_{d,1}(x) f_{c,3}(x) + f_{c,1}(x) f_{d,3}(x) + 2 f_{d,2}(x) f_{c,2}(x) \right) > 0.$$

□

5 Proof of the main result

Assume that (P) has a unique classical solution, then, (6) holds from Theorem 4.6.
Conversely, Assume that (6) holds. From Theorem 4.6, it suffices to show that system (19) has a unique solution such that (25) is satisfies.

The proof is divided in three parts. In the first part, we make explicit the determinant of system (19). Then, in the second part, we show the uniqueness of the solution, to this end, we inverse the determinant with the help of functional calculus. Finally, in the last part, we prove that ψ_1 and ψ_2 have the expected regularity.

5.1 Calculus of the determinant

In this section, we make explicit the determinant. Recall system (19):

$$\begin{cases} \left(P_1^+ + P_1^-\right) M \psi_1 - \left(P_2^+ - P_2^-\right) \psi_2 &= S_1 \\ \left(P_2^+ - P_2^-\right) M \psi_1 - \left(P_3^+ + P_3^-\right) \psi_2 &= S_2. \end{cases}$$

Moreover, we writes the previous system as the matrix equation $\Lambda \Psi = S$, where

$$\Lambda = \begin{pmatrix} M\left(P_1^+ + P_1^-\right) & -\left(P_2^+ - P_2^-\right) \\ M\left(P_2^+ - P_2^-\right) & -\left(P_3^+ + P_3^-\right) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$

To solve system (19), we calculate the determinant of the associated matrix Λ:

$$\begin{aligned} \det(\Lambda) &:= -M\left(P_1^+ + P_1^-\right)\left(P_3^+ + P_3^-\right) + M\left(P_2^+ - P_2^-\right)^2 \\ &= -M\left(P_1^+ P_3^+ + P_1^+ P_3^- + P_1^- P_3^+ + P_1^- P_3^- - \left(P_2^+\right)^2 - \left(P_2^-\right)^2 + 2 P_2^+ P_2^-\right) \\ &= -M\left(P_1^+ P_3^+ - \left(P_2^+\right)^2 + P_1^- P_3^- - \left(P_2^-\right)^2 + P_1^+ P_3^- + P_1^- P_3^+ + 2 P_2^+ P_2^-\right). \end{aligned}$$

We begin to calculate each terms separately. Then, we have

$$\begin{aligned} P_1^+P_3^+ &= k_+^2\left(U_+^{-1}\left(I+e^{dM}\right)^2+V_+^{-1}\left(I-e^{dM}\right)^2\right)\left(U_+^{-1}\left(I-e^{dM}\right)^2+V_+^{-1}\left(I+e^{dM}\right)^2\right)\\ &= k_+^2\left(\left(U_+^{-2}+V_+^{-2}\right)\left(I-e^{2dM}\right)^2+U_+^{-1}V_+^{-1}\left(\left(I+e^{dM}\right)^4+\left(I-e^{dM}\right)^4\right)\right)\end{aligned}$$

and

$$\left(I+e^{dM}\right)^4+\left(I-e^{dM}\right)^4=2\left(I+e^{2dM}\right)^2+8e^{2dM}.$$

Thus

$$P_1^+P_3^+=k_+^2\left(\left(U_+^{-2}+V_+^{-2}\right)\left(I-e^{2dM}\right)^2+2U_+^{-1}V_+^{-1}\left(I+e^{2dM}\right)^2+8U_+^{-1}V_+^{-1}e^{2dM}\right).$$

Moreover, we have

$$\begin{aligned} \left(P_2^+\right)^2 &= k_+^2\left(U_+^{-1}+V_+^{-1}\right)^2\left(I-e^{2dM}\right)^2\\ &= k_+^2\left(U_+^{-2}+V_+^{-2}\right)\left(I-e^{2dM}\right)^2+2k_+^2U_+^{-1}V_+^{-1}\left(I-e^{2dM}\right)^2.\end{aligned}$$

It follows

$$\begin{aligned} P_1^+P_3^+-\left(P_2^+\right)^2 &= k_+^2\left(\left(U_+^{-2}+V_+^{-2}\right)\left(I-e^{2dM}\right)^2+2U_+^{-1}V_+^{-1}\left(I+e^{2dM}\right)^2\right)\\ &\quad +8k_+^2U_+^{-1}V_+^{-1}e^{2dM}\\ &\quad -k_+^2\left(U_+^{-2}+V_+^{-2}\right)\left(I-e^{2dM}\right)^2-2k_+^2U_+^{-1}V_+^{-1}\left(I-e^{2dM}\right)^2\\ &= 2k_+^2U_+^{-1}V_+^{-1}\left(\left(I+e^{2dM}\right)^2-\left(I-e^{2dM}\right)^2+4e^{2dM}\right)\\ &= 2k_+^2U_+^{-1}V_+^{-1}\left(4e^{2dM}+4e^{2dM}\right)\\ &= 16k_+^2U_+^{-1}V_+^{-1}e^{2dM}.\end{aligned}$$

In the same way, replacing respectively k_+, U_+^{-1}, V_+^{-1} and d by k_-, U_-^{-1}, V_-^{-1} and c, we obtain

$$P_1^-P_3^- -\left(P_2^-\right)^2=16k_-^2U_-^{-1}V_-^{-1}e^{2cM}.$$

Thus, the determinant of Λ writes

$$\det(\Lambda)=-M\left(16k_+^2U_+^{-1}V_+^{-1}e^{2dM}+16k_-^2U_-^{-1}V_-^{-1}e^{2cM}+P_1^+P_3^-+P_1^-P_3^++2P_2^+P_2^-\right). \quad (33)$$

Finally, from (32) and (33), we obtain

$$\det(\Lambda)=-Mf(-A). \quad (34)$$

Note that, since $f(-A)\in\mathcal{L}(X)$, it follows that $D(\det(\Lambda))=D(M)$.

5.2 Inversion of the determinant

Let C_1, C_2 two linear operators in X. We note $C_1 \sim C_2$ to means that $C_1 = C_2 + \Sigma$, where Σ is a finite sum of term of type $kM^n e^{\alpha M}$, with $k \in \mathbb{R}$, $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+ \setminus \{0\}$. Note that Σ is a regular term in the sense:

$$\Sigma \in \mathcal{L}(X) \quad \text{with} \quad \Sigma(X) \subset D(M^\infty) := \bigcap_{k \geqslant 0} D(M^k).$$

Since $U_\pm \sim I$ and $V_\pm \sim I$, then setting $W = U_+ U_- V_+ V_- \sim I$, we obtain

$$\begin{cases} WP_1^+ & \sim & 2k_+ I, & WP_1^- & \sim & 2k_- I, \\ WP_2^+ & \sim & 2k_+ I, & WP_2^- & \sim & 2k_- I, \\ WP_3^+ & \sim & 2k_+ I, & WP_3^- & \sim & 2k_- I. \end{cases}$$

Moreover, we have

$$16k_+^2 W^2 U_+^{-1} V_+^{-1} e^{2dM} \sim 0 \quad \text{and} \quad 16k_-^2 W^2 U_-^{-1} V_-^{-1} e^{2cM} \sim 0.$$

We then deduce the following relation

$$\begin{aligned} -M^{-1} W^2 \det(\Lambda) &= 16k_+^2 W^2 U_+^{-1} V_+^{-1} e^{2dM} + 16k_-^2 W^2 U_-^{-1} V_-^{-1} e^{2cM} \\ &\quad + WP_1^+ WP_3^- + WP_1^- WP_3^+ + 2WP_2^+ WP_2^- \\ &\sim 4k_+k_- I + 4k_+k_- I + 8k_+k_- I = 16k_+k_- I. \end{aligned}$$

Thus, we get

$$\det(\Lambda) = -W^{-2} M \left(16k_+k_- I + \sum_{j \in J} k_j M^{n_j} e^{\alpha_j M} \right), \tag{35}$$

where J is a finite set and for all $j \in J$:

$$k_j \in \mathbb{R}, n_j \in \mathbb{N} \quad \text{and} \quad \alpha_j \in \mathbb{R}_+ \setminus \{0\}.$$

From (35), we have

$$\det(\Lambda) = -16k_+k_- W^{-2} M F, \tag{36}$$

where

$$F = I + \sum_{j \in J} \frac{k_j}{16k_+k_-} M^{n_j} e^{\alpha_j M}. \tag{37}$$

For $z \in \mathbb{C} \setminus \mathbb{R}_-$, we set

$$\tilde{f}(z) = 1 + \sum_{j \in J} \frac{k_j}{16k_+k_-} \left(-\sqrt{z}\right)^{n_j} e^{-\alpha_j \sqrt{z}}.$$

Then, $F = \tilde{f}(-A)$ and from (34) and (36), we have

$$f(-A) = -M^{-1} \det(\Lambda) = 16k_+k_- W^{-2} \tilde{f}(-A).$$

Thus, by construction, for $z \in \mathbb{C} \setminus \mathbb{R}_-$, the link between f and $\tilde{f}$ is

$$f(z) = 16k_+k_- u_d^{-2}(z) u_c^{-2}(z) v_d^{-2}(z) v_c^{-2}(z) \tilde{f}(z). \tag{38}$$

Proposition 5.1. The operator $F \in \mathcal{L}(X)$ defined by (37), is invertible with bounded inverse.

Proof. Note that $f, \tilde{f} \in H(S_\theta)$, for a given $\theta \in (0, \pi)$. Moreover, for $z \in \mathbb{C} \setminus \mathbb{R}_-$ and $j \in J$, functions $\dfrac{k_j}{16k_+k_-}(-\sqrt{z})^{n_j}$ are polynomial. Thus, $1 - \tilde{f} \in \mathcal{E}_\infty(S_\theta)$.

From Lemma 4.4 in [31], u_d, u_c, v_d and v_c do not vanish on $\mathbb{C}_+ \setminus \{0\}$. Moreover, due to Lemma 4.8, f does not vanish on $\mathbb{R}_+ \setminus \{0\}$, Thus, we deduce from (38) that $\tilde{f}$ do not vanish on $\mathbb{R}_+ \setminus \{0\}$.

Finally, we apply Lemma 4.7 with $P = -A$ and $G = \tilde{f}$ to obtain that $F = \tilde{f}(-A) \in \mathcal{L}(X)$ is invertible with bounded inverse. □

We are now in position to prove the main result of this section.

Proposition 5.2. The operator $\det(\Lambda)$, defined by (36) is invertible with bounded inverse.

Proof. From Lemma 4.1, U_+, U_-, V_+ and V_- are bounded invertible operators with bounded inverse. So we deduce that W^{-2} is invertible with bounded inverse. Moreover, from (H_2), (36) and Proposition 5.1, we obtain that $\det(\Lambda)$ is invertible with bounded inverse. □

5.3 Regularity

To study the regularity, we need to recall the following technical result from [19], Lemma 5.1.

Lemma 5.3 ([19])**.** Let $V \in \mathcal{L}(X)$ such that $0 \in \rho(I + V)$. Then, there exists $W \in \mathcal{L}(X)$ such that

$$(I + V)^{-1} = I - W,$$

and $W(X) \subset V(X)$. Moreover, if T is a linear operator in X such that $V(X) \subset D(T)$ and for $\psi \in D(T)$, $TV\psi = VT\psi$, then $WT\psi = TW\psi$.

From Proposition 5.2, system (19) has a unique solution (ψ_1, ψ_2). From Theorem 4.6, it remains to prove that

$$\psi_1 \in (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}.$$

To this end, we have to study the regularity of the inverse of the determinant $\det(\Lambda)$.

Lemma 5.4. There exists $R_d \in D(M^\infty)$ such that

$$[\det(\Lambda)]^{-1} = -\frac{1}{16k_+k_-}M^{-1} + R_d.$$

Proof. From (36) and Proposition 5.2, we have

$$[\det(\Lambda)]^{-1} = -\frac{1}{16k_+k_-}M^{-1}W^2F^{-1}.$$

From Lemma 5.3, there exists $R_F \in D(M^\infty)$, such that

$$F^{-1} = I + R_F.$$

Moreover, for $\delta > 0$, we know that

$$e^{\delta M}\psi \in D(M^\infty). \tag{39}$$

Since $W = U_+U_-V_+V_-$, from (39), there exists $R_W \in D(M^\infty)$ such that

$$W^2 = I + R_W.$$

We deduce that there exists $R_d \in D(M^\infty)$, such that

$$[\det(\Lambda)]^{-1} = -\frac{1}{16k_+k_-}M^{-1} + R_d.$$

□

Now, we study the regularity of ψ_1 and ψ_2. We recall that

$$\Lambda\Psi = S,$$

where Λ is invertible from Proposition 5.2. From Lemma 5.3, there exist $R_{U_\pm}, R_{V_\pm} \in D(M^\infty)$, such that

$$U_\pm^{-1} = I + R_{U_\pm} \qquad \text{and} \qquad V_\pm^{-1} = I + R_{V_\pm}. \tag{40}$$

From (39) and (40), there exist $R_i \in D(M^\infty)$, $i = 1, 2, 3, 4$, such that

$$\Lambda = \begin{pmatrix} 2(k_+ + k_-)M + R_1 & -2(k_+ - k_-)I + R_2 \\ 2(k_+ - k_-)M + R_3 & -2(k_+ + k_-)I + R_4 \end{pmatrix}.$$

It follows that there exist $\mathcal{R}_1$, $\mathcal{R}_2 \in D(M^\infty)$, such that

$$\left\{ \begin{array}{lll} \psi_1 & = & \dfrac{(k_+ + k_-)}{8k_+k_-} M^{-1} S_1 - \dfrac{(k_+ - k_-)}{8k_+k_-} M^{-1} S_2 + \mathcal{R}_1 \\ \psi_2 & = & \dfrac{(k_+ - k_-)}{8k_+k_-} S_1 - \dfrac{(k_+ + k_-)}{8k_+k_-} S_2 + \mathcal{R}_2, \end{array} \right. \tag{41}$$

where S_1 is given by (22) and S_2 is given by (24).

Since F_+ is a classical solution of problem (P_+) and F_- is a classical solution of problem (P_-), then from Remark 4.3 and Remark 4.5, we obtain that $\check{S}$ defined by (23) has the following regularity

$$\check{S} \in (D(M), X)_{\frac{1}{p},p}. \tag{42}$$

Moreover, from (7), we have

$$\varphi_1^+, \varphi_1^- \in (D(A), X)_{1+\frac{1}{2p},p} = (D(M), X)_{3+\frac{1}{p},p}$$

and

$$\varphi_2^+, \varphi_2^- \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p} = (D(M), X)_{2+\frac{1}{p},p}.$$

Thus, from (12), (17), Lemma 5.3, Remark 4.3 and Remark 4.5, we deduce that

$$\tilde{\varphi}_1^+, \tilde{\varphi}_1^-, \tilde{\varphi}_3^+, \tilde{\varphi}_3^- \in (D(M), X)_{3+\frac{1}{p},p} \quad \text{and} \quad \tilde{\varphi}_2^+, \tilde{\varphi}_2^-, \tilde{\varphi}_4^+, \tilde{\varphi}_4^- \in (D(M), X)_{2+\frac{1}{p},p}. \tag{43}$$

So, from (22), (24), (39), (42) and (43), we have

$$S_1 \in (D(M), X)_{2+\frac{1}{p},p} \quad \text{and} \quad S_2 \in (D(M), X)_{2+\frac{1}{p},p}. \tag{44}$$

Finally, from (7), (41) and (44), we obtain

$$\psi_1 \in (D(M), X)_{3+\frac{1}{p},p} = (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \psi_2 \in (D(M), X)_{2+\frac{1}{p},p} = (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}.$$

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