Solvability of a transmission problem in L^p-spaces with generalized diffusion equation

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Abstract

We study a transmission problem, in population dynamics, between two juxtaposed habitats. In each habitat, we consider a generalized diffusion equation composed by the Laplace operator and a biharmonic term. We, indeed, allow that the coefficients in front of each term could be negative or null. Using semigroups theory and functional calculus, we give some relation between coefficients to obtain the existence and the uniqueness of the classical solution in L^p-spaces.

Key Words and Phrases: Operational differential equations, functional calculus, analytic semigroups, BIP operators, maximal regularity.

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1 Introduction

In this work, using analytic semigroups theory, we study a transmission problem for a coupled system of generalized diffusion equations in L^p-spaces, with $p \in (1, +\infty)$. We denote by generalized diffusion equation, an equation of the following form

$$k\Delta^2 u - l\Delta u = g,$$

with $k, l \in \mathbb{R}$ and g given. This equation is obtained using the Landau-Ginzburg free energy functional, we refer to [8] or [29] for more details. This work is a natural continuation of the works done in [19] and [35]. Here, we investigate the influence of the Laplace operator and the biharmonic term in the diffusion. In population dynamics, the Laplace operator model the short range diffusion whereas the biharmonic operator represents the long range diffusion. Thus, generalized diffusion is a linear combination of these two operators.

Usually, in most models $k, l > 0$, but in many works like for instance [8], [16], [27], [28] or [29], the authors explain that the biharmonic term plays a stabilizing role if $k > 0$ and a destabilizing role when $k < 0$. This is why, in the present paper, we consider $k \in \mathbb{R} \setminus \{0\}$ and $l \in \mathbb{R}$.

Many works have treated generalized diffusion equations and transmission problems associated to it. For instance, we refer to [3], [6], [8], [10], [20], [21], [22], [23], [28], [29] and [30], for the study of such an equation in population dynamics and to [11], [13], [19] and [35] for transmission problems associated to it. Note that [19] and [35], consider applications in population dynamics wheras [11] and [13], consider applications in plate theory.

We define $\Omega = \Omega_- \cup \Omega_+$, the n-dimensional area, $n \in \mathbb{N} \setminus \{0, 1\}$, constituted by the two juxtaposed habitats $\Omega_- := (a, \gamma) \times \omega$ and $\Omega_+ := (\gamma, b) \times \omega$ with their interface $\Gamma = \{\gamma\} \times \omega$, where $a, \gamma, b \in \mathbb{R}$ with $a < \gamma < b$ and ω being a bounded domain of $\mathbb{R}^{n-1}$.

We investigate the study of the following transmission problem

$$(EQ_{pde}) \left\{ \begin{array}{rcll} k_+\Delta^2 u_+ - l_+\Delta u_+ &=& g_+, & \text{in } \Omega_+ \\ k_-\Delta^2 u_- - l_-\Delta u_- &=& g_-, & \text{in } \Omega_-, \end{array} \right.$$

where $k_\pm \in \mathbb{R} \setminus \{0\}$, $l_\pm \in \mathbb{R}$, $u_\pm \in \Omega_\pm$ are population density and $g_\pm \in L^p(\Omega_\pm)$ are given.

Note that the case $k_\pm, l_\pm > 0$ has been already treated in [19] and the case $l_\pm = 0$ with $k_\pm > 0$ has been already treated in [35]. Thus, in the the present article, the most important new results concern the other cases. Indeed, as in the two previous cases, the key point of this article is the inversion of determinant operator but unlike the two previous cases, the writing of this determinant operator by the functional calculus must take into account the sign of the constants $\frac{l_+}{k_+}$ and $\frac{l_-}{k_-}$. Therefore it is necessary to detail each case and for each of these cases, we give conditions between the different constants $k_\pm$ and $l_\pm$ which make it possible to invert this determinant operator.

Here, we denote by (x,y) the spatial variables with $x \in (a,b)$ and $y \in \omega$. The above equations are supplemented by the following boundary and transmission conditions

$$(BC_{pde}) \begin{cases} (1) \begin{cases} u_-(x,\zeta) &=& 0, \quad x \in (a,\gamma),\ \zeta \in \partial\omega \\ u_+(x,\zeta) &=& 0, \quad x \in (\gamma,b),\ \zeta \in \partial\omega \\ \Delta u_-(x,\zeta) &=& 0, \quad x \in (a,\gamma),\ \zeta \in \partial\omega \\ \Delta u_+(x,\zeta) &=& 0, \quad x \in (\gamma,b),\ \zeta \in \partial\omega \end{cases} \\ (2) \begin{cases} u_-(a,y) &=& \varphi_1^-(y), \quad y \in \omega \\ u_+(b,y) &=& \varphi_1^+(y), \quad y \in \omega \\ \dfrac{\partial u_-}{\partial x}(a,y) &=& \varphi_2^-(y), \quad y \in \omega \\ \dfrac{\partial u_+}{\partial x}(b,y) &=& \varphi_2^+(y), \quad y \in \omega, \end{cases} \end{cases}$$

where $\varphi_1^\pm$ and $\varphi_2^\pm$ are given in suitable spaces, and

$$(TC_{pde}) \begin{cases} u_- &=& u_+ & \text{on } \Gamma \\ \dfrac{\partial u_-}{\partial x} &=& \dfrac{\partial u_+}{\partial x} & \text{on } \Gamma \\ k_-\Delta u_- &=& k_+\Delta u_+ & \text{on } \Gamma \\ \dfrac{\partial}{\partial x}(k_-\Delta u_- - l_- u_-) &=& \dfrac{\partial}{\partial x}(k_+\Delta u_+ - l_+ u_+) & \text{on } \Gamma. \end{cases}$$

In (BC_{pde}), the boundary conditions on the two first lines of (1) means that the individuals could not lie on the boundaries $(a,b) \times \partial\omega$, because, for instance, they die or the edge is impassable. The boundary conditions on the two second lines of (1) mean that there is no dispersal in the normal direction. It follows that the dispersal vanishes on $(a,b) \times \partial\omega$. In (2), the population density and the flux are given, for instance on $\{a\} \times \omega$ and on $\{b\} \times \omega$. This signifies that the habitats are not isolated. Then, in (TC_{pde}), the two first transmission conditions mean the continuity of the density and its flux at the interface, while the two second express, in some sense, the continuity of the dispersal and its flux at the interface Γ.

This article is organized as follows.

In section 2, we give our operational problem. Section 3 is devoted to some recall on BIP operators and real interpolation spaces. In section 4, we give our assumptions and main results. Then, in section 5, we state some preliminary results that will be useful to prove our main result. Finally, section 6, which is composed of three parts, is devoted to the proof of our main result.

2 Operational formulation

Since we have

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \Delta_y u,$$

we set

$$\begin{cases} D(A_0) := \{\psi \in W^{2,p}(\omega) : \psi = 0 \text{ on } \partial\omega\} \\ \forall \psi \in D(A_0), \quad A_0\psi = \Delta_y \psi. \end{cases} \tag{1}$$

Thus, using operator A_0, it follows that each equation of (EQ_{pde}) writes

$$u_\pm^{(4)}(x) + (2A_0 - \frac{l_\pm}{k_\pm} I)u_\pm''(x) + (A_0^2 - \frac{l_\pm}{k_\pm} A_0)u_\pm(x) = f_\pm(x),$$

where $u_\pm(x) := u_\pm(x,\cdot)$, $f_\pm(x) := g_\pm(x,\cdot)/k_\pm$ and $f_- \in L^p(a,\gamma;L^p(\omega))$, $f_+ \in L^p(\gamma,b;L^p(\omega))$ with $p \in (1,+\infty)$.

Moreover, since the boundary conditions (1) of (BC_{pde}) are included in the domain of A_0, problem $(EQ_{pde}) - (BC_{pde}) - (TC_{pde})$ becomes

$$\begin{cases} u_+^{(4)}(x) + (2A_0 - \frac{l_+}{k_+} I)u_+''(x) + (A_0^2 - \frac{l_+}{k_+} A_0)u_+(x) = f_+(x), \quad \text{for a.e. } x \in (\gamma,b) \\ u_-^{(4)}(x) + (2A_0 - \frac{l_-}{k_-} I)u_-''(x) + (A_0^2 - \frac{l_-}{k_-} A_0)u_-(x) = f_-(x), \quad \text{for a.e. } x \in (a,\gamma) \\ u_-(a) = \varphi_1^-, \quad u_+(b) = \varphi_1^+ \\ u_-'(a) = \varphi_2^-, \quad u_+'(b) = \varphi_2^+ \\ u_-(\gamma) = u_+(\gamma) \\ u_-'(\gamma) = u_+'(\gamma) \\ k_-u_-''(\gamma) + k_-A_0u_-(\gamma) = k_+u_+''(\gamma) + k_+A_0u_+(\gamma) \\ k_-u_-^{(3)}(\gamma) + k_-A_0u_-'(\gamma) - l_-u_-'(\gamma) = k_+u_+^{(3)}(\gamma) + k_+A_0u_+'(\gamma) - l_+u_+'(\gamma), \end{cases}$$

Then, we will consider a more general case using $(A, D(A))$, instead of $(A_0, D(A_0))$, with $-A$ a BIP operator of angle $\theta_A \in (0,\pi)$ on a UMD space X, see below for the definitions of BIP operator and UMD spaces, and $f \in L^p(a,b;X)$.

More precisely, setting $r_\pm = \frac{l_\pm}{k_\pm}$, we study the following transmission problem (P):

$$\text{(P)} \begin{cases} (EQ) \begin{cases} u_+^{(4)}(x) + (2A - r_+ I)u_+''(x) + (A^2 - r_+ A)u_+(x) = f_+(x), \quad x \in (\gamma,b) \\ u_-^{(4)}(x) + (2A - r_- I)u_-''(x) + (A^2 - r_- A)u_-(x) = f_-(x), \quad x \in (a,\gamma) \end{cases} \\ (BC) \begin{cases} u_-(a) = \varphi_1^-, \quad u_+(b) = \varphi_1^+ \\ u_-'(a) = \varphi_2^-, \quad u_+'(b) = \varphi_2^+ \end{cases} \\ (TC) \begin{cases} u_-(\gamma) = u_+(\gamma) \\ u_-'(\gamma) = u_+'(\gamma) \\ k_-u_-''(\gamma) + k_-Au_-(\gamma) = k_+u_+''(\gamma) + k_+Au_+(\gamma) \\ k_-u_-^{(3)}(\gamma) + k_-Au_-'(\gamma) - l_-u_-'(\gamma) = k_+u_+^{(3)}(\gamma) + k_+Au_+'(\gamma) - l_+u_+'(\gamma). \end{cases} \end{cases}$$

The transmission conditions (TC) will be divided into

$$(TC1) \begin{cases} u_-(\gamma) = u_+(\gamma) \\ u_-'(\gamma) = u_+'(\gamma), \end{cases}$$

and

$$(TC2) \begin{cases} k_-u_-''(\gamma) + k_-Au_-(\gamma) = k_+u_+''(\gamma) + k_+Au_+(\gamma) \\ k_-u_-^{(3)}(\gamma) + k_-Au_-'(\gamma) - l_-u_-'(\gamma) = k_+u_+^{(3)}(\gamma) + k_+Au_+'(\gamma) - l_+u_+'(\gamma). \end{cases}$$

Note that $(TC2)$ is well defined, see Lemma 3.8 below.

We will search a classical solution of problem (P), that is a solution u such that

$$\left\{\begin{array}{l} u_+ := u_{|(\gamma,b)} \in W^{4,p}(\gamma,b;X) \cap L^p(\gamma,b;D(A^2)), \quad u''_+ \in L^p(\gamma,b;D(A)), \\ u_- := u_{|(a,\gamma)} \in W^{4,p}(a,\gamma;X) \cap L^p(a,\gamma;D(A^2)), \quad u''_- \in L^p(a,\gamma;D(A)), \end{array}\right. \tag{2}$$

and which satisfies $(EQ) - (BC) - (TC)$.

Note that such a solution is not $W^{4,p}(a,b;X)$ but uniquely $W^{4,p}(a,\gamma;X)$ in Ω_- and $W^{4,p}(\gamma,b;X)$ in Ω_+.

3 Definitions and prerequisites

3.1 The class of Bounded Imaginary Powers of operators

Definition 3.1. A Banach space X is a UMD space if and only if for all $p \in (1,+\infty)$, the Hilbert transform is bounded from $L^p(\mathbb{R},X)$ into itself (see [4] and [5]).

Definition 3.2. A closed linear operator T_1 is called sectorial of angle $\alpha \in [0,\pi)$ if

$$i) \quad \sigma(T_1) \subset \overline{S_\alpha},$$

$$ii) \quad \forall\, \alpha' \in (\alpha,\pi), \quad \sup\left\{\|\lambda(\lambda I - T_1)^{-1}\|_{\mathcal{L}(X)} :\ \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}}\right\} < +\infty,$$

where

$$S_\alpha := \left\{\begin{array}{ll} \{z \in \mathbb{C} : z \neq 0 \ \text{ and } \ |\arg(z)| < \alpha\} & \text{if} \quad \alpha \in (0,\pi), \\ (0,+\infty) & \text{if} \quad \alpha = 0, \end{array}\right. \tag{3}$$

see [17], p. 19.

Remark 3.3. From [18], p. 342, we know that any injective sectorial operator T_1 admits imaginary powers T_1^{is}, $s \in \mathbb{R}$, but, in general, T_1^{is} is not bounded.

Definition 3.4. Let $\theta \in [0,\pi)$. We denote by $\mathrm{BIP}(X,\theta)$, the class of sectorial injective operators T_2 such that

$$i) \quad \overline{D(T_2)} = \overline{R(T_2)} = X,$$

$$ii) \quad \forall\, s \in \mathbb{R}, \quad T_2^{is} \in \mathcal{L}(X),$$

$$iii) \quad \exists\, C \geq 1,\ \forall\, s \in \mathbb{R}, \quad ||T_2^{is}||_{\mathcal{L}(X)} \leq Ce^{|s|\theta},$$

see [31], p. 430.

3.2 Interpolation spaces

Here we recall some properties about real interpolation spaces in particular cases.

Definition 3.5. Let $T_3 : D(T_3) \subset X \longrightarrow X$ be a linear operator such that

$$(0,+\infty) \subset \rho(T_3) \quad \text{and} \quad \exists\, C > 0 : \forall\, t > 0, \quad \|t(T_3 - tI)^{-1}\|_{\mathcal{L}(X)} \leqslant C. \tag{4}$$

Let $k \in \mathbb{N} \setminus \{0\}$, $\theta \in (0,1)$ and $q \in [1,+\infty]$. We will use the real interpolation spaces

$$(D(T_3^k), X)_{\theta,q} = (X, D(T_3^k))_{1-\theta,q},$$

defined, for instance, in [24], or in [25].

In particular, for $k=1$, we have the following characterization

$$(D(T_3), X)_{\theta,q} := \left\{ \psi \in X : t \longmapsto t^{1\text{-}\theta} \|T_3(T_3 - tI)^{\text{-}1}\psi\|_X \in L_*^q(0,+\infty) \right\},$$

where $L_*^q(0,+\infty)$ is given by

$$L_*^q(0,+\infty;\mathbb{C}) := \left\{ f \in L^q(0,+\infty) : \left(\int_0^{+\infty} |f(t)|^q \frac{dt}{t} \right)^{1/q} < +\infty \right\}, \quad \text{for } q \in [1,+\infty),$$

and for $q = +\infty$, by

$$L_*^\infty(0,+\infty;\mathbb{C}) := \sup_{t\in(0,+\infty)} |f(t)|,$$

see [9] p. 325, or [15], p. 665, Teorema 3, or section 1.14 of [36], where this space is denoted by $(X, D(T_3))_{1\text{-}\theta,q}$. Note that we can also characterize the space $(D(T_3), X)_{\theta,q}$ taking into account the Osservazione, p. 666, in [15].

We set also, for any $k \in \mathbb{N} \setminus \{0\}$

$$(D(T_3), X)_{k+\theta,q} \;:=\; \left\{ \psi \in D(T_3^k) : T_3^k \psi \in (D(T_3), X)_{\theta,q} \right\},$$

and

$$(X, D(T_3))_{k+\theta,q} \;:=\; \left\{ \psi \in D(T_3^k) : T_3^k \psi \in (X, D(T_3))_{\theta,q} \right\},$$

see [26], definition 3.2, p. 64.

Remark 3.6. The general situation of the real interpolation space $(X_0, X_1)_{\theta,q}$ with X_0, X_1 two Banach spaces such that $X_0 \hookrightarrow X_1$, is described in [24].

Remark 3.7. Note that for T_3 satisfying (4), T_3^k is closed for any $k \in \mathbb{N}\setminus\{0\}$ since $\rho(T_3) \neq \emptyset$; consequently, if $k\theta < 1$, we have

$$(D(T_3^k), X)_{\theta,q} = (X, D(T_3^k))_{1-\theta,q} = (X, D(T_3))_{k-k\theta,q} = (D(T_3), X)_{(k-1)+k\theta,q} \subset D(T_3^{k-1}).$$

For more details see [25], (2.1.13), p. 43 or [15], p. 676, Teorema 6.

We recall the following lemma.

Lemma 3.8 ([15])**.** Let T_3 be a linear operator satisfying (4). Let u such that

$$u \in W^{n,p}(a_1, b_1; X) \cap L^p(a_1, b_1; D(T_3^k)),$$

where $a_1, b_1 \in \mathbb{R}$ with $a_1 < b_1$, $n, k \in \mathbb{N} \setminus \{0\}$ and $p \in (1,+\infty)$. Then for any $j \in \mathbb{N}$ satisfying the Poulsen condition $0 < \frac{1}{p} + j < n$ and $s \in \{a_1, b_1\}$, we have

$$u^{(j)}(s) \in (D(T_3^k), X)_{\frac{j}{n}+\frac{1}{np},p}.$$

This result is proved in [15], p. 678, Teorema 2'.

4 Assumptions and statement of results

4.1 Hypotheses

In all the sequel, $r_+, r_- \in \mathbb{R}$, $k_+k_- > 0$ and A denotes a closed linear operator in X. We assume the following hypotheses:

(H_1) X is a UMD space,

(H_2) $[\min(r_+, r_-, 0), +\infty) \subset \rho(A)$,

(H_3) $-A \in \mathrm{BIP}(X, \theta_A)$ for some $\theta_A \in [0, \pi)$,

(H_4) $-A \in \mathrm{Sect}(0)$.

Remark 4.1.

1. Due to (H_2), if at least one parameter r_+ or r_- is negative or null, then $0 \in \rho(A)$.

2. Operator A_0, defined by (1), satisfies all the previous hypotheses with $X = L^q(\omega)$, $q \in (1, +\infty)$ and $r_\pm \in \rho(A_0)$. From [33], Proposition 3, p. 207, X satisfies (H_1) and taking $A_0 + r_\pm I$ in [14], Theorem 9.15, p. 241 and Lemma 9.17, p. 242, we deduce that A_0 satisfies (H_2). Moreover, (H_3) is satisfied for every $\theta_A \in [0, \pi)$, from [32], Theorem C, p. 166-167. Finally, (H_4) is satisfied thanks to [17], section 8.3, p. 232.

3. In the scalar case, to solve each equation of (EQ), we need to solve the characteristic equations
$$\chi^4 + (2A - r_\pm)\chi^2 + (A^2 - r_\pm A) = 0,$$
thus, in our operational case, we consider the following operators
$$L_- := -\sqrt{-A + r_- I}, \quad L_+ := -\sqrt{-A + r_+ I} \quad \text{and} \quad M := -\sqrt{-A}. \tag{5}$$
Due to (H_2) and (H_3), $-A$, $-A + r_- I$ and $-A + r_+ I$ are sectorial operators, so the existence of L_-, L_+ and M is ensured, see for instance [17], e), p. 25 and [1], Theorem 2.3, p. 69.

4. From [17], Proposition 3.1.9, p. 65, we have $D(L_-) = D(L_+) = D(M)$. Thus, for $n, m \in \mathbb{N}$ and $m \leqslant n$
$$D(L_\pm^n) = D(M^n) = D(L_\pm^m M^{n-m}) = D(M^m L_\pm^{n-m}).$$

5. From [31], Theorem 3, p. 437 and [1], Theorem 2.3, p. 69, assumptions (H_2) and (H_3) imply that $-A + r_\pm I \in \mathrm{BIP}(X, \theta_A)$ and due to [17], Proposition 3.2.1, e), p. 71, that
$$-L_-, -L_+, -M \in \mathrm{BIP}(X, \theta_A/2).$$
Moreover, from [31], Theorem 4, p. 441, we get
$$-(L_- + M), -(L_+ + M) \in \mathrm{BIP}(X, \theta_A/2 + \varepsilon),$$
for any $\varepsilon \in (0, \pi/2 - \theta_A/2)$.
Since we have $0 < \theta_A/2 < \pi/2$, then due to [31], Theorem 2, p. 437, we deduce that L_-, L_+, M, $L_- + M$ and $L_+ + M$ generate bounded analytic semigroups $(e^{xL_-})_{x\geqslant 0}$, $(e^{xL_+})_{x\geqslant 0}$, $(e^{xM})_{x\geqslant 0}$, $(e^{x(L_-+M)})_{x\geqslant 0}$ and $(e^{x(L_++M)})_{x\geqslant 0}$.

6. Using the Dore-Venni sums theorem, see [12], we deduce from (H_1), (H_2) and (H_3) that $0 \in \rho(M) \cap \rho(L_-) \cap \rho(L_+) \cap \rho(L_+ + M) \cap \rho(L_- + M)$.

7. From (5), we deduce that
$$\forall\, \psi \in D(M^2), \quad (L_+^2 - M^2)\psi = r_+ \psi \quad \text{and} \quad (L_-^2 - M^2)\psi = r_- \psi. \tag{6}$$
and also that
$$\forall\, \psi \in D(M), \;\; (L_+ - M)\psi = r_+(L_+ + M)^{-1}\psi \;\text{ and }\; (L_- - M)\psi = r_-(L_- + M)^{-1}\psi. \tag{7}$$

4.2 Main results

To solve our operational problem (P), we introduce two problems:

$$(P_+)\begin{cases} u_+^{(4)}(x) + (2A - r_+ I)u_+''(x) + (A^2 - r_+ A)u_+(x) = f_+(x), \quad \text{for a.e. } x \in (\gamma, b) \\ u_+(\gamma) = \psi_1, \quad u_+(b) = \varphi_1^+ \\ u_+'(\gamma) = \psi_2, \quad u_+'(b) = \varphi_2^+. \end{cases}$$

and

$$(P_-)\begin{cases} u_-^{(4)}(x) + (2A - r_- I)u_-''(x) + (A^2 - r_- A)u_-(x) = f_-(x), \quad \text{for a.e. } x \in (a, \gamma) \\ u_-(a) = \varphi_1^-, \quad u_-(\gamma) = \psi_1 \\ u_-'(a) = \varphi_2^-, \quad u_-'(\gamma) = \psi_2, \end{cases}$$

Remark 4.2. Recall that u is a classical solution of (P) if and only if there exist $\psi_1, \psi_2 \in X$ such that

$$\begin{array}{ll} (i) & u_- \text{ is a classical solution of } (P_-), \\ (ii) & u_+ \text{ is a classical solution of } (P_+), \\ (iii) & u_- \text{ and } u_+ \text{ satisfy } (TC2). \end{array}$$

Therefore, our aim is to state that there exists a unique couple (ψ_1, ψ_2) which satisfies (i), (ii) and (iii).

Theorem 4.3. Let $f_- \in L^p(a, \gamma; X)$ and $f_+ \in L^p(\gamma, b; X)$ with $p \in (1, +\infty)$. Assume that (H_1), (H_2), (H_3), (H_4) hold and $k_+k_- > 0$. Thus

1. when $r_+, r_- \in \mathbb{R} \setminus \{0\}$,
   - if $r_+ > 0$ and $r_- > 0$,
   - if $r_+ < 0$ and $r_- < 0$, such that
   $$(l_+ - l_-)(k_+ - k_-) \geqslant 0,$$
   - if $r_+ > 0$ and $r_- < 0$, such that
   $$-6l_-k_+ + l_+k_+ + l_-k_- \geqslant 0,$$
   - if $r_+ < 0$ and $r_- > 0$, such that
   $$-6l_+k_- + l_+k_+ + l_-k_- \geqslant 0,$$
2. when $r_+ \in \mathbb{R} \setminus \{0\}$ and $r_- = 0$ with $\dfrac{k_-}{k_+} \leqslant 2$,
   - if $r_+ > 0$ such that
   $$r_+ \geqslant \frac{\left(\sqrt{t+1} + \sqrt{t}\right)^2}{t^2} \frac{k_+^2}{4k_-^2}, \quad \text{for } t \in \left(0, \frac{1}{r_+\|A^{-1}\|_{\mathcal{L}(X)}}\right) \text{ fixed},$$
   - if $r_+ < 0$ such that
   $$r_+ \leqslant -\frac{27k_+^2}{64k_-^2},$$

3. when $r_+ = 0$ and $r_- \in \mathbb{R}\setminus\{0\}$ with $\dfrac{k_+}{k_-} \leqslant 2$,

   - if $r_- > 0$ such that
$$r_- \geqslant \frac{\left(\sqrt{t+1}+\sqrt{t}\right)^2}{t^2}\frac{k_-^2}{4k_+^2}, \quad \text{for } t \in \left(0, \frac{1}{r_-\|A^{-1}\|_{\mathcal{L}(X)}}\right) \text{ fixed},$$
   - if $r_- < 0$ such that
$$r_- \leqslant -\frac{27k_-^2}{64k_+^2},$$

then, there exists a unique classical solution u, of the transmission problem (P) if and only if
$$\varphi_1^+, \varphi_1^- \in (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_2^+, \varphi_2^- \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{8}$$

Remark 4.4. Since the third case is the symmetric of the second one, replacing k_+ by k_- and l_+ by l_-, the proof if exactly the same. Thus, we omit it.

Remark 4.5. If $A = A_0$, to satisfy the first condition set in the second or the third case of Theorem 4.3, since $\|A^{-1}\|_{\mathcal{L}(X)} \geqslant \frac{1}{C_\omega}$, where C_ω is the Poincaré constant, it suffices to take ω sufficiently small because the more ω is small, the more C_ω is large, see for instance [7], Corollary 2.2, p. 95 and Corollary 2.3, p. 96, or [2], Remark 3, p. 15.

As a consequence of the previous Theorem, we state the following corollary.

Corollary 4.6. Let $n \geqslant 2$, $f_+ \in L^p(\Omega_+)$ and $f_- \in L^p(\Omega_-)$ with $p \in (1, +\infty)$ and $p > n$. Assume that ω is a bounded open set of $\mathbb{R}^{n-1}$ with C^2-boundary. Let $k_+, k_-, l_+ > 0$ and $l_- < 0$ with $k_+ = k_-$. Then, there exists a unique solution u of (P_{pde}), such that we have $u_- \in W^{4,p}(\Omega_-)$ and $u_+ \in W^{4,p}(\Omega_+)$, if and only if
$$\begin{cases} \varphi_1^\pm, \varphi_2^\pm \in W^{2,p}(\omega) \cap W_0^{1,p}(\omega) \\ \Delta\varphi_1^\pm, \in W^{2-\frac{1}{p},p}(\omega) \cap W_0^{1,p}(\omega) \\ \Delta\varphi_2^\pm \in W^{1-\frac{1}{p},p}(\omega) \cap W_0^{1,p}(\omega). \end{cases}$$

The proof is quite similar to the one stated in [19], Corollary 1, p. 2941, or in [20], Corollary 2.7, p. 357. Thus we omit it.

5 Preliminary results

In all the sequel, we set
$$c = \gamma - a > 0 \quad \text{and} \quad d = b - \gamma > 0.$$

From Remark 4.2, to solve problem (P) we must first study problems (P_+) and (P_-). To this end, we need the following invertibility result obtained in [20] and [34].

Lemma 5.1 ([20] and [34])**.** Assume that (H_1), (H_2), (H_3) and (H_4) hold. Then operators $U_\pm$, $V_\pm \in \mathcal{L}(X)$ defined by

$$\left\{\begin{array}{lcl} U_+ & := & \left\{\begin{array}{ll} I - e^{d(L_+ + M)} - \dfrac{1}{r_+}(L_+ + M)^2\left(e^{dM} - e^{dL_+}\right), & \text{if } r_+ \in \mathbb{R}\setminus\{0\} \\ I - e^{2dM} + 2dMe^{dM}, & \text{if } r_+ = 0 \end{array}\right. \\ V_+ & := & \left\{\begin{array}{ll} I - e^{d(L_+ + M)} + \dfrac{1}{r_+}(L_+ + M)^2\left(e^{dM} - e^{dL_+}\right), & \text{if } r_- \in \mathbb{R}\setminus\{0\} \\ I - e^{2dM} - 2dMe^{dM}, & \text{if } r_+ = 0 \end{array}\right. \\ U_- & := & \left\{\begin{array}{ll} I - e^{c(L_- + M)} - \dfrac{1}{r_-}(L_- + M)^2\left(e^{cM} - e^{cL_-}\right), & \text{if } r_- \in \mathbb{R}\setminus\{0\} \\ I - e^{2cM} + 2cMe^{cM}, & \text{if } r_- = 0 \end{array}\right. \\ V_- & := & \left\{\begin{array}{ll} I - e^{c(L_- + M)} + \dfrac{1}{r_-}(L_- + M)^2\left(e^{cM} - e^{cL_-}\right), & \text{if } r_- \in \mathbb{R}\setminus\{0\} \\ I - e^{2cM} - 2cMe^{cM}, & \text{if } r_- = 0, \end{array}\right. \end{array}\right. \tag{9}$$

are invertible with bounded inverses.

From Remark 4.1, statement 4, $U_\pm$ and $V_\pm$ are well defined. For a detailed proof, see [20], Proposition 5.4 with $k = r_\pm$ and [34], Proposition 4.5, p. 645.

5.1 Transmission system

5.1.1 First case

Assume that $r_\pm \in \mathbb{R}\setminus\{0\}$. We set

$$\left\{\begin{array}{lcl} P_1^+ & = & k_+(L_+ + M)\left(U_+^{-1}(I + e^{dM})(I - e^{dL_+}) + V_+^{-1}(I - e^{dM})(I + e^{dL_+})\right) \\ P_2^+ & = & k_+(L_+ + M)\left(U_+^{-1}(I - e^{dM})(I - e^{dL_+}) + V_+^{-1}(I + e^{dM})(I + e^{dL_+})\right) \\ P_3^+ & = & k_+(L_+ + M)L_+\left(U_+^{-1}(I + e^{dM})(I + e^{dL_+}) + V_+^{-1}(I - e^{dM})(I - e^{dL_+})\right), \end{array}\right. \tag{10}$$

and similarly

$$\left\{\begin{array}{lcl} P_1^- & = & k_-(L_- + M)\left(U_-^{-1}(I + e^{cM})(I - e^{cL_-}) + V_-^{-1}(I - e^{cM})(I + e^{cL_-})\right) \\ P_2^- & = & k_-(L_- + M)\left(U_-^{-1}(I - e^{cM})(I - e^{cL_-}) + V_-^{-1}(I + e^{cM})(I + e^{cL_-})\right) \\ P_3^- & = & k_-(L_- + M)L_-\left(U_-^{-1}(I + e^{cM})(I + e^{cL_-}) + V_-^{-1}(I - e^{cM})(I - e^{cL_-})\right). \end{array}\right. \tag{11}$$

Moreover, we note

$$\begin{array}{rcl} S_1 & = & k_+(L_+ + M)\left(U_+^{-1}(I - e^{dL_+})\tilde{\varphi}_2{}^+ + V_+^{-1}(I + e^{dL_+})\tilde{\varphi}_4{}^+\right) \\ & & -k_-(L_- + M)\left(U_-^{-1}(I - e^{cL_-})\tilde{\varphi}_2{}^- + V_-^{-1}(I + e^{cL_-})\tilde{\varphi}_4{}^-\right), \end{array} \tag{12}$$

and

$$\begin{array}{rcl} S_2 & = & -k_+(L_+ + M)\left(U_+^{-1}(I + e^{dM})\tilde{\varphi}_1{}^+ + V_+^{-1}(I - e^{dM})\tilde{\varphi}_3{}^+\right) \\ & & -k_-(L_- + M)\left(U_-^{-1}(I + e^{cM})\tilde{\varphi}_1{}^- + V_-^{-1}(I - e^{cM})\tilde{\varphi}_3{}^-\right) - 2M^{-1}R_1, \end{array} \tag{13}$$

with

$$R_1 = -k_+F_+'''(\gamma) + k_+M^2F_+'(\gamma) + l_+F_+'(\gamma) + k_-F_-'''(\gamma) - k_-M^2F_-'(\gamma) - l_-F_-'(\gamma), \tag{14}$$

where F_+ is the unique classical solution of problem

$$\begin{cases} u_+^{(4)}(x) + (2A - r_+ I)u_+''(x) + (A^2 - r_+ A)u_+(x) = f_+(x), & \text{for a.e. } x \in (\gamma, b) \\ u_+(\gamma) = u_+(b) = u_+''(\gamma) = u_+''(b) = 0, \end{cases} \tag{15}$$

and F_- is the unique classical solution of problem

$$\begin{cases} u_-^{(4)}(x) + (2A - r_- I)u_-''(x) + (A^2 - r_- A)u_-(x) = f_-(x), & \text{for a.e. } x \in (a, \gamma) \\ u_-(a) = u_-(\gamma) = u_-''(a) = u_+''(\gamma) = 0. \end{cases} \tag{16}$$

For an explicit representation formula of the solution of both previous problems, we refer to [20], Theorem 2.2, p. 355-356.

Remark 5.2. Since $F_\pm$ is a classical solution of (15), respectively (16), from Lemma 3.8, it follows that, for $j = 0, 1, 2, 3$ and $s = a$, γ or b

$$F_\pm^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{p},p}.$$

Now, with our notations, we recall a useful result of [19], Theorem 4.6, p. 2945. This result is proved for $r_\pm > 0$ but clearly remains true for $r_\pm < 0$.

Theorem 5.3 ([19])**.** Let $f_- \in L^p(a, \gamma; X)$ and $f_+ \in L^p(\gamma, b; X)$, with $p \in (1, +\infty)$. Assume that (H_1), (H_2), (H_3) and (H_4) hold. Then, the transmission problem (P) has a unique classical solution if and only if the data $\varphi_1^+, \varphi_1^-, \varphi_2^+, \varphi_2^-$ satisfy (8) and system

$$\begin{cases} \left(P_1^- - P_1^+\right) M\psi_1 & + & \left(P_2^+ + P_2^-\right)\psi_2 & = & S_1 \\ \left(P_3^+ + P_3^-\right)\psi_1 & + & \left(P_1^- - P_1^+\right)\psi_2 & = & S_2, \end{cases} \tag{17}$$

has a unique solution (ψ_1, ψ_2) such that

$$(\psi_1, \psi_2) \in (D(A), X)_{1+\frac{1}{2p},p} \times (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{18}$$

5.1.2 Second case

Now, assume that $r_- = 0$; then $l_- = 0$ and the transmission conditions $(TC2)$ becomes

$$(TC2')\begin{cases} k_- u_-''(\gamma) + k_- A u_-(\gamma) & = & k_+ u_+''(\gamma) + k_+ A u_+(\gamma) \\ k_- u_-^{(3)}(\gamma) + k_- A u_-'(\gamma) & = & k_+ u_+^{(3)}(\gamma) + k_+ A u_+'(\gamma) - l_+ u_+'(\gamma). \end{cases} \tag{19}$$

Our aim here is to establish a similar result to the previous one. To this end, we set

$$\begin{cases} Q_1^- & = & k_- \left(U_-^{-1} + V_-^{-1}\right)\left(I - e^{2cM}\right) \\ Q_2^- & = & k_- \left(U_-^{-1}\left(I - e^{cM}\right)^2 + V_-^{-1}\left(I + e^{cM}\right)^2\right) \\ Q_3^- & = & k_- \left(U_-^{-1}\left(I + e^{cM}\right)^2 + V_-^{-1}\left(I - e^{cM}\right)^2\right), \end{cases} \tag{20}$$

Moreover, we note

$$\begin{aligned} S_3 \; = \; & 2k_- M\left(U_-^{-1}\left(I - e^{cM}\right)\tilde{\varphi_2}^- + V_-^{-1}\left(I + e^{cM}\right)\tilde{\varphi_4}^-\right) \\ & -k_+(L_+ + M)\left(U_+^{-1}\left(I - e^{dL_+}\right)\tilde{\varphi_2}^+ + V_+^{-1}\left(I + e^{dL_+}\right)\tilde{\varphi_4}^+\right), \end{aligned} \tag{21}$$

and

$$S_4 = -2k_- M\left(U_-^{-1}\left(I+e^{cM}\right)\tilde{\varphi}_2^- + V_-^{-1}\left(I-e^{cM}\right)\tilde{\varphi}_4^-\right) -k_+(L_+ + M)\left(U_+^{-1}\left(I+e^{dM}\right)\tilde{\varphi}_1^+ + V_+^{-1}\left(I-e^{dM}\right)\tilde{\varphi}_3^+\right) + 2M^{-1}R_2, \quad (22)$$

with

$$R_2 = -k_-\tilde{F}_-'''(\gamma) + k_- M^2 \tilde{F}_-'(\gamma) + k_+ F_+'''(\gamma) - k_+ M^2 F_+'(\gamma) - l_+ F_+'(\gamma), \quad (23)$$

where $\tilde{F}_-$ is the classical solution of problem

$$\begin{cases} u_-^{(4)}(x) + 2Au_-''(x) + A^2 u_-(x) = f_-(x), & \text{a.e. } x \in (a,\gamma) \\ u_-(a) = u_-(\gamma) = u_-''(a) = u_-''(\gamma) = 0. \end{cases} \quad (24)$$

Remark 5.4. Since $\tilde{F}_-$ is a classical solution of (16), as in Remark 5.2, from Lemma 3.8, it follows that, for $j = 0, 1, 2, 3$ and $s = a$, γ or b

$$\tilde{F}_-^{(j)}(s) \in (D(M), X)_{3-j+\frac{1}{p},p}.$$

Theorem 5.5. Let $f_- \in L^p(a,\gamma;X)$ and $f_+ \in L^p(\gamma,b;X)$, with $p \in (1,+\infty)$. Assume that (H_1), (H_2), (H_3) and (H_4) hold. Then, problem (P) has a unique classical solution if and only if the data φ_1^+, φ_1^-, φ_2^+, φ_2^- satisfy (8) and system

$$\begin{cases} \left(P_1^+ - 2MQ_1^-\right) M\psi_1 - \left(P_2^+ + 2MQ_2^-\right)\psi_2 = S_3 \\ \left(P_3^+ + 2MQ_3^-\right)\psi_1 + \left(2MQ_1^- - P_1^+\right)\psi_2 = S_4, \end{cases} \quad (25)$$

has a unique solution (ψ_1, ψ_2) satisfying (18).

Proof. We follow the same steps than the proof of Theorem 4.6, p. 2945 in [19], we only point out the key points. From [20], Theorem 2.5, statement 2, there exists a unique classical solution u_+ of (P_+) if and only if

$$\varphi_1^+, \psi_1 \in (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_2^+, \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \quad (26)$$

Recall that, from Remark 3.7, we have

$$(D(A), X)_{1+\frac{1}{2p},p} = (D(M), X)_{3+\frac{1}{p},p} \quad \text{and} \quad (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p} = (D(M), X)_{2+\frac{1}{p},p}. \quad (27)$$

This solution is explicitly given in [19], Proposition 2, p. 2943-2944, from which we deduce that

$$k_+\left(u_+''(\gamma) - M^2 u_+(\gamma)\right) = l_+\left(I - e^{dL_+}\right)\alpha_2^+ + l_+\left(I + e^{dL_+}\right)\alpha_4^+,$$

and

$$k_+\left(u_+^{(3)}(\gamma) - M^2 u_+'(\gamma)\right) - l_+ u_+'(\gamma) = -l_+ M\left(I + e^{dM}\right)\alpha_1^+ - l_+ M\left(I - e^{dM}\right)\alpha_3^+ + k_+ F_+'''(\gamma) - k_+ M^2 F_+'(\gamma) - l_+ F_+'(\gamma),$$

where

$$\begin{cases} \alpha_1^+ = \dfrac{1}{2r_+}(L_+ + M)U_+^{-1}\left[L_+(I+e^{dL_+})\psi_1 - (I - e^{dL_+})\psi_2 + \tilde{\varphi}_1^+\right] \\ \alpha_2^+ = -\dfrac{1}{2r_+}(L_+ + M)U_+^{-1}\left[M(I+e^{dM})\psi_1 - (I - e^{dM})\psi_2 + \tilde{\varphi}_2^+\right] \\ \alpha_3^+ = \dfrac{1}{2r_+}(L_+ + M)V_+^{-1}\left[L_+(I-e^{dL_+})\psi_1 - (I + e^{dL_+})\psi_2 + \tilde{\varphi}_3^+\right] \\ \alpha_4^+ = -\dfrac{1}{2r_+}(L_+ + M)V_+^{-1}\left[M(I-e^{dM})\psi_1 - (I + e^{dM})\psi_2 + \tilde{\varphi}_4^+\right], \end{cases} \quad (28)$$

with

$$\begin{cases} \tilde{\varphi_1}^+ &= -L_+\left(I+e^{dL_+}\right)\varphi_1^+ + \left(I-e^{dL_+}\right)\left(F'_+(b)+F'_+(\gamma)-\varphi_2^+\right) \\ \tilde{\varphi_2}^+ &= -M\left(I+e^{dM}\right)\varphi_1^+ + \left(I-e^{dM}\right)\left(F'_+(b)+F'_+(\gamma)-\varphi_2^+\right) \\ \tilde{\varphi_3}^+ &= L_+\left(I-e^{dL_+}\right)\varphi_1^+ - \left(I+e^{dL_+}\right)\left(F'_+(b)-F'_+(\gamma)-\varphi_2^+\right) \\ \tilde{\varphi_4}^+ &= M\left(I-e^{dM}\right)\varphi_1^+ - \left(I+e^{dM}\right)\left(F'_+(b)-F'_+(\gamma)-\varphi_2^+\right), \end{cases} \tag{29}$$

and F_+ is the unique classical solution of problem (15).

In the same way, from [34], Theorem 2.8, statement 2, p. 637, there exists a unique classical solution u_- of problem (P_-) if and only if

$$\varphi_1^-, \psi_1 \in (D(A), X)_{1+\frac{1}{2p},p} \quad \text{and} \quad \varphi_2^-, \psi_2 \in (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \tag{30}$$

Note that from (27), we have

$$\varphi_1^-, \psi_1 \in (D(M), X)_{3+\frac{1}{p},p} \quad \text{and} \quad \varphi_2^-, \psi_2 \in (D(M), X)_{2+\frac{1}{p},p}.$$

Moreover, this solution, given in [34], Proposition 4.1, p. 640, is explicitly written in [35], Proposition 4.2, from which it follows that

$$k_-\left(u''_-(\gamma) - M^2 u_-(\gamma)\right) = -k_-\left(2M\left(I-e^{cM}\right)\alpha_2^- - 2M\left(I+e^{cM}\right)\alpha_4^-\right),$$

and

$$\begin{aligned} k_-\left(u_-^{(3)}(\gamma) - M^2 u'_-(\gamma)\right) &= k_-\left(2M^2\left(I+e^{cM}\right)\alpha_2^- - 2M^2\left(I-e^{cM}\right)\alpha_4^-\right) \\ &\quad + k_- F_-^{(3)}(\gamma) - k_- M^2 F'_-(\gamma), \end{aligned}$$

where

$$\begin{cases} \alpha_1^- &:= -\frac{1}{2}U_-^{-1}\left[\left(I+(I+cM)e^{cM}\right)\psi_1 - ce^{cM}\psi_2 + \tilde{\varphi}_1^-\right] \\ \alpha_2^- &:= \frac{1}{2}U_-^{-1}\left[\left(I+e^{cM}\right)M\psi_1 + \left(I-e^{cM}\right)\psi_2 + \tilde{\varphi}_2^-\right] \\ \alpha_3^- &:= \frac{1}{2}V_-^{-1}\left[\left(I-(I+cM)e^{cM}\right)\psi_1 + ce^{cM}\psi_2 + \tilde{\varphi}_3^-\right] \\ \alpha_4^- &:= -\frac{1}{2}V_-^{-1}\left[\left(I-e^{cM}\right)M\psi_1 + \left(I+e^{cM}\right)\psi_2 + \tilde{\varphi}_4^-\right], \end{cases} \tag{31}$$

with

$$\begin{cases} \tilde{\varphi_1}^- &:= -\left(I+e^{cM}\right)\varphi_1^- - ce^{cM}\left(M\varphi_1^- + \varphi_2^- - \tilde{F}'_-(a) - \tilde{F}'_-(\gamma)\right) \\ \tilde{\varphi_2}^- &:= -M\left(I+e^{cM}\right)\varphi_1^- + \left(I-e^{cM}\right)\left(\varphi_2^- - \tilde{F}'_-(a) - \tilde{F}'_-(\gamma)\right) \\ \tilde{\varphi_3}^- &:= \left(I-e^{cM}\right)\varphi_1^- - ce^{cM}\left(M\varphi_1^- + \varphi_2^- - \tilde{F}'_-(a) + \tilde{F}'_-(\gamma)\right) \\ \tilde{\varphi_4}^- &:= M\left(I-e^{cM}\right)\varphi_1^- - \left(I+e^{cM}\right)\left(\varphi_2^- - \tilde{F}'_-(a) + \tilde{F}'_-(\gamma)\right). \end{cases} \tag{32}$$

Note that due to (26), (27), (28) and (29), respectively to (27), (30), (31) and (32), we deduce that

$$\alpha_i^\pm \in D(M), \quad \text{for } i = 1,2,3,4 \quad \text{and} \quad \alpha_2^-, \alpha_4^- \in D(M^2).$$

Thus, system $(TC2')$, given by (19), writes

$$\begin{cases} -2k_-M\left(\left(I-e^{cM}\right)\alpha_2^- - \left(I+e^{cM}\right)\alpha_4^-\right) = l_+\left(\left(I-e^{dL_+}\right)\alpha_2^+ + \left(I+e^{dL_+}\right)\alpha_4^+\right) \\ 2k_-M^2\left(\left(I+e^{cM}\right)\alpha_2^- - \left(I-e^{cM}\right)\alpha_4^-\right) = -l_+M\left(\left(I+e^{dM}\right)\alpha_1^+ + \left(I-e^{dM}\right)\alpha_3^+\right)+R_2, \end{cases}$$

where R_2 is given by (23). Thus, it follows that the previous system gives

$$\left\{\begin{array}{rcl} -2k_-U_-^{-1}M\left(I-e^{cM}\right)\left[\left(I+e^{cM}\right)M\psi_1+\left(I-e^{cM}\right)\psi_2+\tilde{\varphi}_2^-\right] & & \\ -2k_-V_-^{-1}M\left(I+e^{cM}\right)\left[\left(I-e^{cM}\right)M\psi_1+\left(I+e^{cM}\right)\psi_2+\tilde{\varphi}_4^-\right] & & \\ +k_+(L_++M)U_+^{-1}\left(I-e^{dL_+}\right)\left[M(I+e^{dM})\psi_1-(I-e^{dM})\psi_2+\tilde{\varphi}_2^+\right] & & \\ +k_+(L_++M)V_+^{-1}\left(I+e^{dL_+}\right)\left[M(I-e^{dM})\psi_1-(I+e^{dM})\psi_2+\tilde{\varphi}_4^+\right] & = & 0 \\ & & \\ 2k_-MU_-^{-1}\left(I+e^{cM}\right)\left[\left(I+e^{cM}\right)M\psi_1+\left(I-e^{cM}\right)\psi_2+\tilde{\varphi}_2^-\right] & & \\ +2k_-MV_-^{-1}\left(I-e^{cM}\right)\left[\left(I-e^{cM}\right)M\psi_1+\left(I+e^{cM}\right)\psi_2+\tilde{\varphi}_4^-\right] & & \\ +k_+(L_++M)U_+^{-1}\left(I+e^{dM}\right)\left[L_+(I+e^{dL_+})\psi_1-(I-e^{dL_+})\psi_2+\tilde{\varphi}_1^+\right] & & \\ +k_+(L_++M)V_+^{-1}\left(I-e^{dM}\right)\left[L_+(I-e^{dL_+})\psi_1-(I+e^{dL_+})\psi_2+\tilde{\varphi}_3^+\right] & = & 2M^{-1}R_2, \end{array}\right.$$

Finally, using (10), (20), (21), (22) and (23), we obtain that the previous system writes as system (25).

Conversely, if we assume that (8) holds and system (25) has a unique solution (ψ_1, ψ_2) satisfying (18), then considering $u_\pm$ the unique classical solution of $(P_\pm)$, we obtain that u is the unique classical solution of (P). □

5.2 Functional calculus

In this section, by using functional calculus, we rewrite the operators defined in (9), (10), (11) and (20), to inverse the determinant operator of system (17) and system (25).

To this end, we recall some classical notations. For $\theta \in (0, \pi)$, we denote by $H(S_\theta)$ the space of holomorphic functions on S_θ (defined by (3)) with values in $\mathbb{C}$. Moreover, we consider the following subspace of $H(S_\theta)$:

$$\mathcal{E}_\infty(S_\theta) := \left\{f \in H(S_\theta) : f = O(|z|^{-s}) \ (|z| \to +\infty) \text{ for some } s > 0\right\}.$$

In other words, $\mathcal{E}_\infty(S_\theta)$ is the space of polynomially decreasing holomorphic functions at $+\infty$. Let T be an invertible sectorial operator of angle $\theta_T \in (0, \pi)$. If $f \in \mathcal{E}_\infty(S_\theta)$, with $\theta \in (\theta_T, \pi)$, then we can define, by functional calculus, $f(T) \in \mathcal{L}(X)$, see [17], p. 45.

Then, we recall a useful result from [20], Lemma 5.3, p. 370.

Lemma 5.6 ([20])**.** Let P be an invertible sectorial operator in X with angle θ, for all $\theta \in (0, \pi)$. Let $G \in H(S_\theta)$, for some $\theta \in (0, \pi)$, such that

(i) $1 - G \in \mathcal{E}_\infty(S_\theta)$,

(ii) $G(x) \neq 0$ for any $x \in \mathbb{R}_+ \setminus \{0\}$.

Then, $G(P) \in \mathcal{L}(X)$, is invertible with bounded inverse.

Let $r \in \mathbb{R}$, $r_m = \max(-r, 0)$, $\delta > 0$ and $z \in \mathbb{C} \setminus (-\infty, r_m]$. We set

$$\left\{\begin{array}{l} u_{\delta,r}(z) = \left\{\begin{array}{ll} 1 - e^{-\delta(\sqrt{z+r}+\sqrt{z})} - \dfrac{1}{r}(\sqrt{z+r}+\sqrt{z})^2\left(e^{-\delta\sqrt{z}} - e^{-\delta\sqrt{z+r}}\right), & \text{if } r \in \mathbb{R}\setminus\{0\} \\ 1 - e^{-2\delta\sqrt{z}} - 2\delta\sqrt{z}e^{-\delta\sqrt{z}}, & \text{if } r = 0 \end{array}\right. \\ \\ v_{\delta,r}(z) = \left\{\begin{array}{ll} 1 - e^{-\delta(\sqrt{z+r}+\sqrt{z})} + \dfrac{1}{r}(\sqrt{z+r}+\sqrt{z})^2\left(e^{-\delta\sqrt{z}} - e^{-\delta\sqrt{z+r}}\right), & \text{if } r \in \mathbb{R}\setminus\{0\} \\ 1 - e^{-2\delta\sqrt{z}} + 2\delta\sqrt{z}e^{-\delta\sqrt{z}}, & \text{if } r = 0 \end{array}\right. \end{array}\right.$$

and when $u_{\delta,r}(z) \neq 0$, $v_{\delta,r}(z) \neq 0$, we note

$$f_{\delta,r,1}(z) = \left\{\begin{array}{ll} \left(\sqrt{z+r}+\sqrt{z}\right)\sqrt{z+r}\, u_{\delta,r}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)\left(1+e^{-\delta\sqrt{z+r}}\right) & \\ +\left(\sqrt{z+r}+\sqrt{z}\right)\sqrt{z+r}\, v_{d,r}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)\left(1-e^{-\delta\sqrt{z+r}}\right), & \text{if } r \in \mathbb{R}\setminus\{0\} \\ \left(u_{\delta,0}^{-1}(z) + v_{\delta,0}^{-1}(z)\right)\left(1-e^{-2\delta\sqrt{z}}\right), & \text{if } r = 0, \end{array}\right.$$

$$f_{\delta,r,2}(z) = \left\{\begin{array}{ll} -\left(\sqrt{z+r}+\sqrt{z}\right) u_{\delta,r}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)\left(1-e^{-\delta\sqrt{z+r}}\right) & \\ -\left(\sqrt{z+r}+\sqrt{z}\right) v_{\delta,r}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)\left(1+e^{-\delta\sqrt{z+r}}\right), & \text{if } r \in \mathbb{R}\setminus\{0\} \\ u_{\delta,0}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)^2 + v_{\delta,0}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)^2, & \text{if } r = 0, \end{array}\right.$$

and

$$f_{\delta,r,3}(z) = \left\{\begin{array}{ll} -\left(\sqrt{z+r}+\sqrt{z}\right) u_{\delta,r}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)\left(1-e^{-\delta\sqrt{z+r}}\right) & \\ -\left(\sqrt{z+r}+\sqrt{z}\right) v_{\delta,r}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)\left(1+e^{-\delta\sqrt{z+r}}\right), & \text{if } r \in \mathbb{R}\setminus\{0\} \\ u_{\delta,0}^{-1}(z)\left(1+e^{-\delta\sqrt{z}}\right)^2 + v_{\delta,0}^{-1}(z)\left(1-e^{-\delta\sqrt{z}}\right)^2, & \text{if } r = 0 \end{array}\right.$$

Remark 5.7. Note that, from (H_2) and (H_3), if $r_\pm \neq 0$, we have

$$\left\{\begin{array}{rrr} u_{c,r_-}(-A) = U_-, & u_{d,r_+}(-A) = U_+, & v_{c,r_-}(-A) = V_-, \\ v_{d,r_+}(-A) = V_+, & k_- f_{c,r_-,1}(-A) = P_1^-, & k_+ f_{d,r_+,1}(-A) = P_1^+, \\ k_- f_{c,r_-,2}(-A) = P_2^-, & k_+ f_{d,r_+,2}(-A) = P_2^+, & k_- f_{c,r_-,3}(-A) = P_3^-, \\ k_+ f_{d,r_+,3}(-A) = P_3^+, & & \end{array}\right.$$

and if $r_- = 0$, we obtain

$$\left\{\begin{array}{rrr} u_{c,0}(-A) = U_-, & v_{c,0}(-A) = V_-, & k_- f_{c,0,1}(-A) = Q_1^-, \\ k_- f_{c,0,2}(-A) = Q_2^-, & k_- f_{c,0,3}(-A) = Q_3^-. & \end{array}\right.$$

Remark 5.8. Let $\delta > 0$, $r \in \mathbb{R}$ and $x \in (r_m, +\infty)$. Then, when $r = 0$, we have

$$1 - e^{-2\delta\sqrt{x}} \pm 2\delta\sqrt{x}e^{-\delta\sqrt{x}} = 2e^{-\delta\sqrt{x}}\left(\sinh(\delta\sqrt{x}) \pm \delta\sqrt{x}\right) > 0,$$

and from [20], Lemma 5.2, p. 369, it clear that $u_{\delta,r}(x) > 0$ and $v_{\delta,r}(x) > 0$. Thus, when $r \neq 0$, we deduce that

$$f_{\delta,r,1}(x) > 0 \quad \text{and} \quad f_{\delta,r,2}(x), f_{\delta,r,3}(x) < 0,$$

and when $r = 0$, we obtain

$$f_{\delta,0,1}(x), f_{\delta,0,2}(x), f_{\delta,0,3}(x) > 0.$$

Moreover, for $z \in \mathbb{C} \setminus (-\infty, r_m]$ and $r \in \mathbb{R} \setminus \{0\}$, we define

$$\begin{aligned} g_{\delta,r}(z) &= -\sqrt{z+r}\left(\left(1-e^{-2\delta(\sqrt{z+r}+\sqrt{z})}\right)^2 - \frac{1}{r^2}(\sqrt{z+r}+\sqrt{z})^4\left(e^{-2\delta\sqrt{z}} - e^{-2\delta\sqrt{z+r}}\right)^2\right) \\ &\quad +\sqrt{z}\left(\left(1-e^{-\delta(\sqrt{z+r}+\sqrt{z})}\right)^2 + \frac{1}{r}(\sqrt{z+r}+\sqrt{z})^2\left(e^{-\delta\sqrt{z}} - e^{-\delta\sqrt{z+r}}\right)^2\right)^2, \end{aligned}$$

and for $r = 0$, we set

$$g_{\delta,0}(z) = (1+\sqrt{z})\left(1-e^{-2\delta\sqrt{z}}\right)^4 + 4\left(1-e^{-2\delta\sqrt{z}}\right)^2 e^{-2\delta\sqrt{z}} - 16\,\delta^2 z\, e^{-4\delta\sqrt{z}}$$

Lemma 5.9. Let $\delta > 0$ and $x \in (r_m, +\infty)$. Thus, if $r \in \mathbb{R} \setminus \{0\}$, then $g_{\delta,r}(x) < 0$ and if $r = 0$, then $g_{\delta,0}(x) > 0$.

Proof. Let $\delta > 0$ and $r \in \mathbb{R} \setminus \{0\}$. For all $x \in (r_m, +\infty)$, from [19], Lemma 4.4, p. 2950, we obtain the result.

Now, consider that $r = 0$. Then

$$\begin{aligned} g_{\delta,0}(x) &= (1+\sqrt{x})\left(1-e^{-2\delta\sqrt{x}}\right)^4 + 4e^{-2\delta\sqrt{x}}\left(\left(1-e^{-2\delta\sqrt{x}}\right)^2 - 4\,\delta^2 x\, e^{-2\delta\sqrt{x}}\right) \\ &= (1+\sqrt{x})\left(1-e^{-2\delta\sqrt{x}}\right)^4 \\ &\quad +4e^{-2\delta\sqrt{x}}\left(1-e^{-2\delta\sqrt{x}} - 2\,\delta\sqrt{x}\,e^{-\delta\sqrt{x}}\right)\left(1-e^{-2\delta\sqrt{x}} + 2\,\delta\sqrt{x}\,e^{-\delta\sqrt{x}}\right). \end{aligned}$$

Hence, since $\delta, x > 0$, we have

$$\begin{aligned} 1-e^{-2\delta\sqrt{x}} - 2\,\delta\sqrt{x}\,e^{-\delta\sqrt{x}} &= e^{-\delta\sqrt{x}}\left(e^{\delta\sqrt{x}} - e^{-\delta\sqrt{x}} - 2\,\delta\sqrt{x}\right) \\ &= 2e^{-\delta\sqrt{x}}\left(\sinh(\delta\sqrt{x}) - \delta\sqrt{x}\right) > 0. \end{aligned}$$

Finally, we deduce that $g_{\delta,0} > 0$. □

6 Proof of the main results

In both cases, assume that problem (P) has a unique classical solution; thus, from Theorem 5.3, respectively Theorem 5.5, (8) holds. Conversely, assume that (8) holds, then due to Theorem 5.3, respectively Theorem 5.5, we have to prove that system (17), respectively system (25), has a unique solution such that (18) holds.

The proof is divided in three parts for both cases. First, we will make explicit, in the first case, the determinant of system (17) and in the second case, the determinant of system (25). Then, in the two cases, we will show the uniqueness of the solution. To this end, we will invert the determinant thanks to functional calculus. Finally, we will prove, in all cases, that ψ_1 and ψ_2 have the expected regularity.

6.1 Calculus of the determinant

6.1.1 First case

Here, we consider $r_+, r_- \in \mathbb{R} \setminus \{0\}$. We have to make explicit the determinant of system (17) that we recall here

$$\begin{cases} \left(P_1^- - P_1^+\right) M\psi_1 &+ \quad \left(P_2^+ + P_2^-\right)\psi_2 &= \quad S_1 \\ \left(P_3^+ + P_3^-\right)\psi_1 &+ \quad \left(P_1^- - P_1^+\right)\psi_2 &= \quad S_2. \end{cases}$$

We write the previous system as a matrix equation $\Lambda_1\Psi = S$, where

$$\Lambda_1 = \begin{pmatrix} \left(P_1^- - P_1^+\right)M & \left(P_2^+ + P_2^-\right) \\ \left(P_3^+ + P_3^-\right) & \left(P_1^- - P_1^+\right) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$

To solve system (17), we will study the determinant

$$\det(\Lambda_1) := M\left(P_1^- - P_1^+\right)^2 - \left(P_2^+ + P_2^-\right)\left(P_3^+ + P_3^-\right),$$

of the matrix Λ_1. Thus, we set

$$\det(\Lambda_1) = D_1^+ + D_1^- + D_2, \tag{33}$$

where

$$\begin{cases} D_1^+ &= M\left(P_1^+\right)^2 - P_3^+P_2^+ \\ D_1^- &= M\left(P_1^-\right)^2 - P_3^-P_2^- \\ D_2 &= -P_3^+P_2^- - P_3^-P_2^+ - 2MP_1^+P_1^-. \end{cases}$$

Then, we recall the result of [19] (Lemma 5.1, p. 2953), describing the determinant.

Lemma 6.1 ([19])**.** We have

1. $D_1^+ = -4k_+^2(L_+ + M)^2U_+^{-2}V_+^{-2}D^+$, with

$$\begin{aligned} D^+ &= L_+\left(\left(I - e^{2d(L_+ + M)}\right)^2 - \frac{1}{r_+^2}(L_+ + M)^4\left(e^{2dM} - e^{2dL_+}\right)^2\right) \\ &\quad -M\left(\left(I - e^{d(L_+ + M)}\right)^2 + \frac{1}{r_+^2}(L_+ + M)^2\left(e^{dM} - e^{dL_+}\right)^2\right)^2. \end{aligned}$$

2. $D_1^- = -4k_-^2(L_- + M)^2U_-^{-2}V_-^{-2}D^-$, with

$$\begin{aligned} D^- &= L_-\left(\left(I - e^{2c(L_- + M)}\right)^2 - \frac{1}{r_-^2}(L_- + M)^4\left(e^{2cM} - e^{2cL_-}\right)^2\right) \\ &\quad -M\left(\left(I - e^{c(L_- + M)}\right)^2 + \frac{1}{r_-^2}(L_- + M)^2\left(e^{cM} - e^{cL_-}\right)^2\right)^2. \end{aligned}$$

6.1.2 Second case

Here, we consider $r_+ \in \mathbb{R}\setminus\{0\}$ and $r_- = 0$. As previously, we make explicit the determinant of system (25) that we recall here

$$\begin{cases} \left(P_1^+ - 2MQ_1^-\right)M\psi_1 &- \left(P_2^+ + 2MQ_2^-\right)\psi_2 &= S_3 \\ \left(P_3^+ + 2MQ_3^-\right)\psi_1 &+ \left(2MQ_1^- - P_1^+\right)\psi_2 &= S_4, \end{cases}$$

We write this system as a matrix equation $\Lambda_2\Psi = \tilde{S}$, where

$$\Lambda_2 = \begin{pmatrix} \left(P_1^+ - 2MQ_1^-\right)M & -\left(P_2^+ + 2MQ_2^-\right) \\ \left(P_3^+ + 2MQ_3^-\right) & \left(2MQ_1^- - P_1^+\right) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{and} \quad \tilde{S} = \begin{pmatrix} S_3 \\ S_4 \end{pmatrix}.$$

To solve system (17), we will study the determinant

$$\det(\Lambda_2) := -M\left(P_1^+ - 2MQ_1^-\right)^2 + \left(P_3^+ + 2MQ_3^-\right)\left(P_2^+ + 2MQ_2^-\right),$$

of the matrix Λ_2. We set

$$\det(\Lambda_2) = D_3^+ + D_3^- + D_4, \tag{34}$$

where

$$D_3^+ = P_2^+P_3^+ - M\left(P_1^+\right)^2 \quad \text{and} \quad D_3^- = 4M^2\left(Q_2^-Q_3^- - M\left(Q_1^-\right)^2\right),$$

with

$$D_4 = 2M\left(P_3^+Q_2^- + P_2^+Q_3^- + MP_1^+Q_1^-\right).$$

Lemma 6.2. We have

1. $D_3^+ = k_+^2(L_+ + M)^2U_+^{-2}V_+^{-2}D_0^+$, with

$$\begin{aligned} D_0^+ &= L_+\left(\left(I - e^{2d(L_++M)}\right)^2 - \frac{1}{r_+^2}(L_+ + M)^4\left(e^{2dM} - e^{2dL_+}\right)^2\right) \\ &\quad -M\left(\left(I - e^{d(L_++M)}\right)^2 + \frac{1}{r_+^2}(L_+ + M)^2\left(e^{dM} - e^{dL_+}\right)^2\right)^2. \end{aligned}$$

2. $D_3^- = 16k_-^2M^2U_-^{-2}V_-^{-2}D_0^-$, with

$$D_0^- = (I - M)\left(I - e^{2cM}\right)^4 + 4\left(I - e^{2cM}\right)^2e^{2cM} - 16c^2M^2e^{4cM}.$$

Proof.

1. We have

$$P_2^+P_3^+ = k_+^2(L_+ + M)^2L_+U_+^{-2}V_+^{-2}D'_+,$$

where

$$\begin{aligned} D'_+ &= \left(U_+^2 + V_+^2\right)(I - e^{2dM})(I - e^{2dL_+}) \\ &\quad +U_+V_+\left((I + e^{dM})^2(I + e^{dL_+})^2 + (I - e^{dM})^2(I - e^{dL_+})^2\right) \\ &= \left(U_+^2 + V_+^2\right)\left[\left(I + e^{d(L_++M)}\right)^2 - \left(e^{dM} + e^{dL_+}\right)^2\right] \\ &\quad +2U_+V_+\left[\left(I + e^{d(L_++M)}\right)^2 + \left(e^{dM} + e^{dL_+}\right)^2\right] \\ &= (U_+ + V_+)^2\left(I + e^{d(L_++M)}\right)^2 - (V_+ - U_+)^2\left(e^{dM} + e^{dL_+}\right)^2. \end{aligned}$$

Moreover, from (9), we obtain that

$$U_+ + V_+ = 2\left(I - e^{d(L_++M)}\right) \quad \text{and} \quad V_+ - U_+ = \frac{2}{r_+}(L_+ + M)^2\left(e^{dM} - e^{dL_+}\right). \tag{35}$$

Then

$$\begin{aligned} D'_+ &= 4\left(I - e^{d(L_++M)}\right)^2\left(I + e^{d(L_++M)}\right)^2 \\ &\quad -\frac{4}{r_+^2}(L_+ + M)^4\left(e^{dM} - e^{dL_+}\right)^2\left(e^{dM} + e^{dL_+}\right)^2 \\ &= 4\left(I - e^{2d(L_++M)}\right)^2 - \frac{4}{r_+^2}(L_+ + M)^4\left(e^{2dM} - e^{2dL_+}\right)^2. \end{aligned}$$

Furthermore, we have

$$M\left(P_1^+\right)^2 = k_+^2(L_+ + M)^2 M U_+^{-2} V_+^{-2} D_+'',$$

where

$$\begin{aligned} D_+'' &= \left(V_+(I+e^{dM})(I-e^{dL_+}) + U_+(I-e^{dM})(I+e^{dL_+})\right)^2 \\ &= \left[(U_+ + V_+)\left(I - e^{d(L_+ + M)}\right) + (V_+ - U_+)\left(e^{dM} - e^{dL_+}\right)\right]^2, \end{aligned}$$

and due to (35), it follows that

$$D_+'' = \left[2\left(I - e^{d(L_+ + M)}\right)^2 + \frac{2}{r_+}(L_+ + M)^2\left(e^{dM} - e^{dL_+}\right)^2\right]^2.$$

Finally, we deduce that

$$\begin{aligned} D_3^+ &= P_2^+ P_3^+ - M\left(P_1^+\right)^2 \\ &= k_+^2(L_+ + M)^2 U_+^{-2} V_+^{-2}\left(L_+ D_+' - M D_+''\right), \end{aligned}$$

and setting $D_0^+ = L_+ D_+' - M D_+''$, we obtain the expected result.

2. We have

$$Q_2^- Q_3^- = k_-^2 U_-^{-2} V_-^{-2} D_-',$$

where

$$\begin{aligned} D_-' &= \left(V_-\left(I - e^{cM}\right)^2 + U_-\left(I + e^{cM}\right)^2\right)\left(V_-\left(I + e^{cM}\right)^2 + U_-\left(I - e^{cM}\right)^2\right) \\ &= \left(U_-^2 + V_-^2\right)\left(I - e^{2cM}\right)^2 + 2U_- V_-\left(I - e^{2cM}\right)^2 + 16 U_- V_- e^{2cM}, \end{aligned}$$

and

$$\begin{aligned} M\left(Q_1^-\right)^2 &= k_-^2 M U_-^{-2} V_-^{-2}\left(U_- + V_-\right)^2\left(I - e^{2cM}\right)^2 \\ &= k_-^2 U_-^{-2} V_-^{-2}\left[M\left(U_-^2 + V_-^2\right)\left(I - e^{2cM}\right)^2 + 2M U_- V_-\left(I - e^{2cM}\right)^2\right]. \end{aligned}$$

Thus

$$Q_2^- Q_3^- - M\left(Q_1^-\right)^2 = k_-^2 U_-^{-2} V_-^{-2} D_-'',$$

where

$$\begin{aligned} D_-'' &= \left(I - M\right)\left(U_-^2 + V_-^2\right)\left(I - e^{2cM}\right)^2 + 2\left(I - M\right) U_- V_-\left(I - e^{2cM}\right)^2 \\ &\quad + 16 U_- V_- e^{2cM} \\ &= \left(I - M\right)\left(U_- + V_-\right)^2\left(I - e^{2cM}\right)^2 + 16 U_- V_- e^{2cM}. \end{aligned}$$

Moreover, from (9), we obtain that

$$U_- + V_- = 2\left(I - e^{2cM}\right) \quad \text{and} \quad U_- V_- = \left(I - e^{2cM}\right)^2 - 4c^2 M^2 e^{2cM}.$$

Then

$$D''_- = 4\,(I-M)\left(I-e^{2cM}\right)^4 + 16\left(I-e^{2cM}\right)^2 e^{2cM} - 64c^2M^2e^{4cM}.$$

Therefore, it follows that

$$\begin{array}{rcl} D_3^- & = & 4M^2\left(Q_2^-Q_3^- - M\left(Q_1^-\right)^2\right) \\ & = & 16k_-^2 M^2 U_-^{-2} V_-^{-2} D_0^-, \end{array}$$

where $D_0^- = \dfrac{1}{4}\,D''_-$.

□

6.2 Inversion of the determinant

6.2.1 First case

Here, we consider $r_+, r_- \in \mathbb{R}\setminus\{0\}$. Let $r = \max(-r_+, -r_-, 0) \geqslant 0$. By using functional calculus, we prove that the determinant of system (17), given by (33), is invertible with bounded inverse. Due to Lemma 6.1 and the definition of D_2, we obtain:

$$D_1^+ = g_1^+(-A), \quad D_1^- = g_1^-(-A) \quad \text{and} \quad D_2 = g_2(-A),$$

where, for $z \in \mathbb{C}\setminus\mathbb{R}_-$, we have set

$$\left\{\begin{array}{rcl} g_1^+(z) & = & 4k_+^2(\sqrt{z+r_+}+\sqrt{z})^2 u_{d,r_+}^{-2}(z) v_{d,r_+}^{-2}(z) g_{d,r_+}(z) \\ g_1^-(z) & = & 4k_-^2(\sqrt{z+r_-}+\sqrt{z})^2 u_{c,r_-}^{-2}(z) v_{c,r_-}^{-2}(z) g_{c,r_-}(z) \\ g_2(z) & = & k_+f_{d,r_+,1}(z)k_-f_{c,r_-,3}(z) + k_-f_{c,r_-,1}(z)k_+f_{d,r_+,3}(z) \\ & & -2\sqrt{z}\,k_+f_{d,r_+,2}(z)k_-f_{c,r_-,2}(z), \end{array}\right.$$

with $u_{\delta,r}$, $v_{\delta,r}$, $g_{\delta,r}$ and $f_{\delta,r,i}$ the complex functions defined in section 5.2. Thus

$$\det(\Lambda_1) = D_1^+ + D_1^- + D_2 = f_1(-A), \tag{36}$$

with $f_1 = g_1^+ + g_1^- + g_2$. Note that, for some $\theta \in (0,\pi)$, we have $f_1 \in H(S_\theta)$ and due to Remark 5.8 and Lemma 5.9, for $x > 0$, we have

$$f_1(x) = g_1^+(x) + g_1^-(x) + g_2(x) < 0. \tag{37}$$

Let C_1, C_2 be two linear operators in X. We denote by $C_1 \sim C_2$ the equality $C_1 = C_2 + \Sigma$, where Σ is a finite sum of terms of type $kL_+^l L_-^m M^n e^{\alpha L_+} e^{\beta L_-} e^{\delta M}$, where $k \in \mathbb{R}$; $l,m,n \in \mathbb{N}$; $\alpha,\beta,\delta \in \mathbb{R}_+$ with $\alpha+\beta+\delta \neq 0$. Note that Σ is a regular term in the sense:

$$\Sigma \in \mathcal{L}(X) \quad \text{with} \quad \Sigma(X) \subset D(M^\infty) := \bigcap_{k\geqslant 0} D(M^k).$$

Since we have $U_\pm \sim I, V_\pm \sim I$, then by setting $W = U_-U_+V_-V_+ \sim I$, we deduce that

$$\left\{\begin{array}{rclrcl} WP_1^+ & \sim & 2k_+(L_++M), & WP_1^- & \sim & 2k_-(L_-+M) \\ WP_2^+ & \sim & 2k_+(L_++M), & WP_2^- & \sim & 2k_-(L_-+M) \\ WP_3^+ & \sim & 2k_+(L_++M)L_+, & WP_3^- & \sim & 2k_-(L_-+M)L_-. \end{array}\right.$$

Thus

$$\begin{aligned} W^2\det(\Lambda_1) &= M\left(WP_1^+\right)^2-\left(WP_2^+WP_3^+\right)+M\left(WP_1^-\right)^2-\left(WP_2^-WP_3^-\right)\\ &\quad -\left(WP_2^-WP_3^+ + WP_2^+WP_3^- + 2MWP_1^+WP_1^-\right)\\ &\sim -4k_+^2(L_++M)^2(L_+-M)-4k_-^2(L_-+M)^2(L_--M)\\ &\quad -4k_+k_-(L_++M)(L_-+M)(L_++L_-+2M). \end{aligned}$$

From (7), we have

$$\begin{aligned} -W^2\det(\Lambda_1) &\sim 4k_+^2r_+(L_++M)+4k_-^2r_-(L_-+M)\\ &\quad +4k_+k_-(L_++M)(L_-+M)(L_++L_-+2M)\\ &\sim 4k_+l_+(L_++M)+4k_-l_-(L_-+M)\\ &\quad +4k_+k_-(L_++M)(L_-+M)(L_++L_-+2M) \end{aligned}$$

Hence, we note

$$B_1 = 4k_+l_+(L_++M)+4k_-l_-(L_-+M)+4k_+k_-(L_++M)(L_-+M)(L_++L_-+2M).$$

Thus, we obtain

$$\det(\Lambda_1) = -W^{-2}\left(B_1+\sum_{j\in J}k_jL_+^{l_j}L_-^{m_j}M^{n_j}e^{\alpha_jL_+}e^{\beta_jL_-}e^{\delta_jM}\right), \tag{38}$$

where J is a finite set and for any $j\in J$:

$$k_j\in\mathbb{R};\ l_j,m_j,n_j\in\mathbb{N},\ \alpha_j,\beta_j,\delta_j\in\mathbb{R}_+ \quad \text{with} \quad \alpha_j+\beta_j+\delta_j\neq 0.$$

We set

$$B_2 = I+\frac{l_+}{k_-}(L_-+M)^{-1}(L_++L_-+2M)^{-1}+\frac{l_-}{k_+}(L_++M)^{-1}(L_++L_-+2M)^{-1}$$

such that

$$B_1 = 4k_+k_-(L_++M)(L_-+M)(L_++L_-+2M)B_2.$$

Proposition 6.3. Assume that (H_1), (H_2), (H_3), (H_4) hold and $k_+k_->0$. If one of the following assumptions holds

- $\dfrac{l_+}{k_-}>0$ and $\dfrac{l_-}{k_+}>0$,
- $\dfrac{l_+}{k_-}<0$ and $\dfrac{l_-}{k_+}<0$, such that

$$(l_+-l_-)(k_+-k_-)\geqslant 0, \tag{39}$$

- $\dfrac{l_+}{k_-}>0$ and $\dfrac{l_-}{k_+}<0$, such that

$$-6l_-k_++l_+k_++l_-k_-\geqslant 0, \tag{40}$$

- $\dfrac{l_+}{k_-} < 0$ and $\dfrac{l_-}{k_+} > 0$, such that

$$-6l_+k_- + l_+k_+ + l_-k_- \geqslant 0, \tag{41}$$

then, $b_2(x) > 0$, for $x > r \geqslant 0$ and operator B_1, defined above, is invertible with bounded inverse.

Remark 6.4. Since $k_+k_- > 0$, then we have the following equivalences

$$\frac{l_+}{k_-} > 0 \Longleftrightarrow r_+ > 0 \quad \text{and} \quad \frac{l_-}{k_+} > 0 \Longleftrightarrow r_- > 0.$$

Proof. From (H_2) and (H_3), since $k_+k_- \neq 0$, it is clear that

$$0 \in \rho\left(4k_+k_-(L_+ + M)(L_- + M)(L_+ + L_- + 2M)\right).$$

Thus, it remains to prove that $0 \in \rho(B_2)$. To this end, we use Lemma 5.6.

Let $z \in \mathbb{C} \setminus (-\infty, r]$. We set

$$\begin{aligned} b_2(z) &= 1 + \frac{l_+}{k_-}\frac{1}{(\sqrt{z+r_-}+\sqrt{z})(\sqrt{z+r_+}+\sqrt{z+r_-}+2\sqrt{z})} \\ &\quad + \frac{l_-}{k_+}\frac{1}{(\sqrt{z+r_+}+\sqrt{z})(\sqrt{z+r_+}+\sqrt{z+r_-}+2\sqrt{z})}, \end{aligned} \tag{42}$$

hence $b_2(-A) = B_2$. Then, for all $x > r \geqslant 0$, it follows

$$\begin{aligned} b_2(x) &= 1 + \frac{l_+}{k_-}\frac{1}{(\sqrt{x+r_-}+\sqrt{x})(\sqrt{x+r_+}+\sqrt{x+r_-}+2\sqrt{x})} \\ &\quad + \frac{l_-}{k_+}\frac{1}{(\sqrt{x+r_+}+\sqrt{x})(\sqrt{x+r_+}+\sqrt{x+r_-}+2\sqrt{x})} \end{aligned}$$

Our aim is to prove that $b_2(x) > 0$, for all $x > r$. To this end, we set

$$y = x - r > 0,$$

hence

$$\begin{aligned} b_2(y+r) &= 1 + \frac{l_+}{k_-}\frac{1}{(\sqrt{y+r+r_-}+\sqrt{y+r})(\sqrt{y+r+r_+}+\sqrt{y+r+r_-}+2\sqrt{y+r})} \\ &\quad + \frac{l_-}{k_+}\frac{1}{(\sqrt{y+r+r_+}+\sqrt{y+r})(\sqrt{y+r+r_+}+\sqrt{y+r+r_-}+2\sqrt{y+r})} \\ &= 1 + \frac{1}{(\sqrt{y+r+r_+}+\sqrt{y+r+r_-}+2\sqrt{y+r})}b_3(y), \end{aligned}$$

where

$$b_3(y) = \frac{\frac{l_+}{k_-}}{(\sqrt{y+r+r_-}+\sqrt{y+r})} + \frac{\frac{l_-}{k_+}}{(\sqrt{y+r+r_+}+\sqrt{y+r})}.$$

Then

$$b_3'(y) = \frac{-\frac{l_+}{k_-}\left(\frac{1}{2\sqrt{y+r+r_-}}+\frac{1}{2\sqrt{y+r}}\right)}{(\sqrt{y+r+r_-}+\sqrt{y+r})^2} + \frac{-\frac{l_-}{k_+}\left(\frac{1}{2\sqrt{y+r+r_+}}+\frac{1}{2\sqrt{y+r}}\right)}{(\sqrt{y+r+r_+}+\sqrt{y+r})^2}$$

and

$$b_2'(y+r) = \frac{-\left(\frac{1}{2\sqrt{y+r+r_+}}+\frac{1}{2\sqrt{y+r+r_-}}+\frac{1}{\sqrt{y+r}}\right)}{(\sqrt{y+r+r_+}+\sqrt{y+r+r_-}+2\sqrt{y+r})^2}b_3(y) + \frac{1}{(\sqrt{y+r+r_+}+\sqrt{y+r+r_-}+2\sqrt{y+r})}b_3'(y),$$

Now, we have to study the following fourth cases.

1. If $\frac{l_+}{k_-} > 0$ and $\frac{l_-}{k_+} > 0$, then it is clear that $b_3 > 0$ and $b_2 > 0$.

2. If $\frac{l_+}{k_-} < 0$ and $\frac{l_-}{k_+} < 0$, then $b_3' > 0$ and $b_2' > 0$. Thus $b_2(y+r) > b_2(r)$ where

$$b_2(r) = 1 + \frac{1}{(\sqrt{r+r_+}+2\sqrt{r})}b_3(0) > 1 + \frac{1}{2\sqrt{r}}b_3(0),$$

with

$$b_3(0) = \frac{\frac{l_+}{k_-}}{\sqrt{r+r_-}+\sqrt{r}} + \frac{\frac{l_-}{k_+}}{\sqrt{r+r_+}+\sqrt{r}}.$$

Since $\sqrt{r+r_+} > 0$ and $\sqrt{r+r_-} > 0$, it follows that

$$\frac{\frac{l_+}{k_-}}{\sqrt{r+r_-}+\sqrt{r}} > \frac{\frac{l_+}{k_-}}{\sqrt{r}} \quad \text{and} \quad \frac{\frac{l_-}{k_+}}{\sqrt{r+r_+}+\sqrt{r}} > \frac{\frac{l_-}{k_+}}{\sqrt{r}},$$

hence

$$b_3(0) > \frac{1}{\sqrt{r}}\left(\frac{l_+}{k_-}+\frac{l_-}{k_+}\right).$$

Thus, we obtain

$$b_2(r) > 1 + \frac{1}{2r}\left(\frac{l_+}{k_-}+\frac{l_-}{k_+}\right).$$

Moreover, we have

$$1 + \frac{1}{2r}\left(\frac{l_+}{k_-}+\frac{l_-}{k_+}\right) \geqslant 0 \Longleftrightarrow 2r \geqslant -\left(\frac{l_+}{k_-}+\frac{l_-}{k_+}\right),$$

where

$$-\left(\frac{l_+}{k_-}+\frac{l_-}{k_+}\right) = \begin{cases} -r_-\left(\frac{l_+}{l_-}+\frac{k_-}{k_+}\right), & \text{if } r = -r_- \\ -r_+\left(\frac{k_+}{k_-}+\frac{l_-}{l_+}\right), & \text{if } r = -r_+. \end{cases}$$

Thus, we obtain that

$$2r \geqslant -\left(\frac{l_+}{k_-}+\frac{l_-}{k_+}\right) \Longleftrightarrow \begin{cases} l_+k_+ + l_-k_- - 2l_-k_+ \geqslant 0, & \text{if } r = -r_- \\ l_+k_+ + l_-k_- - 2l_+k_- \geqslant 0, & \text{if } r = -r_+. \end{cases}$$

Furthermore, since $k_+k_- > 0$, if $r = -r_-$, then $-\frac{l_-}{k_-} \geqslant -\frac{l_+}{k_+}$, hence $-l_-k_+ \geqslant -l_+k_-$ and if $r = -r_+$, then $-\frac{l_+}{k_+} \geqslant -\frac{l_-}{k_-}$, hence $-l_+k_- \geqslant -l_-k_+$. It follows that

$$\begin{cases} l_+k_+ + l_-k_- - 2l_-k_+ \geqslant l_+k_+ + l_-k_- - l_+k_- - l_-k_+, & \text{if } r = -r_- \\ l_+k_+ + l_-k_- - 2l_+k_- \geqslant l_+k_+ + l_-k_- - l_+k_- - l_-k_+, & \text{if } r = -r_+. \end{cases}$$

Finally, if (39) holds, then we obtain $b_2 > 0$.

3. If $\frac{l_+}{k_-} > 0$ and $\frac{l_-}{k_+} < 0$, then since $k_+ k_- > 0$, we have

$$\frac{l_+}{k_-} > 0 \Longleftrightarrow \frac{l_+}{k_+}\frac{k_+}{k_-}k_-^2 > 0 \Longleftrightarrow r_+ > 0 \quad \text{and} \quad \frac{l_-}{k_+} < 0 \Longleftrightarrow \frac{l_-}{k_-}\frac{k_-}{k_+}k_+^2 < 0 \Longleftrightarrow r_- < 0.$$

Thus $r = -r_- > 0$, $r_+ > 0$ and

$$b_3(y) = \frac{\frac{l_+}{k_-}}{(\sqrt{y} + \sqrt{y+r})} + \frac{\frac{l_-}{k_+}}{(\sqrt{y+r+r_+} + \sqrt{y+r})}.$$

Since $\frac{l_+}{k_-} > 0$ and $\sqrt{y} < \sqrt{y+r}$ it follows that

$$\frac{\frac{l_+}{k_-}}{\sqrt{y} + \sqrt{y+r}} > \frac{\frac{l_+}{k_-}}{2\sqrt{y+r}}.$$

In the same way, since $\frac{l_-}{k_+} < 0$ and $\sqrt{y+r+r_+} > \sqrt{y+r}$, we deduce that

$$\frac{\frac{l_-}{k_+}}{\sqrt{y+r+r_+} + \sqrt{y+r}} > \frac{\frac{l_-}{k_+}}{2\sqrt{y+r}},$$

hence

$$b_3(y) > \frac{1}{2\sqrt{y+r}}\left(\frac{l_+}{k_-} + \frac{l_-}{k_+}\right).$$

If $\frac{l_+}{k_-} + \frac{l_-}{k_+} > 0$, then $b_3 > 0$ and $b_2 > 0$. If $\frac{l_+}{k_-} + \frac{l_-}{k_+} < 0$, then we have

$$\begin{aligned} b_2(y+r) &= 1 + \frac{1}{(\sqrt{y+r+r_+} + \sqrt{y} + 2\sqrt{y+r})} b_3(y) \\ &> 1 + \frac{1}{3\sqrt{y+r}} b_3(y) \\ &> 1 + \frac{1}{6(y+r)}\left(\frac{l_+}{k_-} + \frac{l_-}{k_+}\right). \end{aligned}$$

Moreover, we have

$$1 + \frac{1}{6(y+r)}\left(\frac{l_+}{k_-} + \frac{l_-}{k_+}\right) \geqslant 0 \Longleftrightarrow 6(y+r) + \frac{l_+}{k_-} + \frac{l_-}{k_+} \geqslant 0.$$

It is obvious that

$$6(y+r) + \left(\frac{l_+}{k_-} + \frac{l_-}{k_+}\right) \geqslant 6r + \frac{l_+}{k_-} + \frac{l_-}{k_+},$$

thus, since $k_+ k_- > 0$ and here $r = -r_- = -\frac{l_-}{k_-}$, we deduce that the previous inequality becomes

$$-6\frac{l_-}{k_-} + \frac{l_+}{k_-} + \frac{l_-}{k_+} \geqslant 0 \Longleftrightarrow -6l_- k_+ + l_+ k_+ + l_- k_- \geqslant 0.$$

Finally, since $k_+ k_- > 0$, if (40) holds, then $b_2 > 0$.

4. If $\frac{l_+}{k_-} < 0$ and $\frac{l_-}{k_+} > 0$, then here $r = -r_+$ and in the same way than previously, if (41) holds, then $b_2 > 0$.

Since $r = \max(-r_+, -r_-, 0) \geqslant 0$ and due to (H_2) and (H_3), we deduce that operator $-A - rI \in \text{BIP}(X, \theta_A)$ with $0 \in \rho(-A - rI)$. Thus, considering $\tilde{b_2}(z) = b_2(z + r)$, with $z + r \in \mathbb{C} \setminus \mathbb{R}_-$, it follows that $\tilde{b_2}(-A - rI) = B_2$. Moreover, for a given $\theta \in (0, \pi)$, it is clear that $1 - b_2, 1 - \tilde{b_2} \in \mathcal{E}_\infty$. Finally, applying Lemma 5.6 with $G = \tilde{b_2}$ and $P = -A - rI$, we deduce the result. $\square$

Due to (38) and Proposition 6.3, it follows that

$$\det(\Lambda_1) = -W^{-2} B_1 F_1, \tag{43}$$

where

$$F_1 = I + \sum_{j \in J} k_j B_1^{-1} L_+^{l_j} L_-^{m_j} M^{n_j} e^{\alpha_j L_+} e^{\beta_j L_-} e^{\delta_j M}. \tag{44}$$

For $z \in \mathbb{C} \setminus (-\infty, r]$, we set

$$b_1(z) = -4k_+k_-(\sqrt{z + r_+} + \sqrt{z})(\sqrt{z + r_-} + \sqrt{z})(\sqrt{z + r_+} + \sqrt{z + r_-} + 2\sqrt{z}) b_2(z), \tag{45}$$

where b_2 is given by (42) and

$$\tilde{f}_1(z) = 1 + \sum_{j \in J} k_j b_1(z)^{-1} \left(-\sqrt{z + r_+}\right)^{l_j} \left(-\sqrt{z + r_-}\right)^{m_j} \left(-\sqrt{z}\right)^{n_j} e^{-\alpha_j \sqrt{z + r_+}} e^{-\beta_j \sqrt{z + r_-}} e^{-\delta_j \sqrt{z}}.$$

Then, due to (H_2) and (H_3), we have $B_1 = b_1(-A)$ and $F_1 = \tilde{f}_1(-A)$. Moreover, from (36) and (43), we obtain

$$f_1(-A) = \det(\Lambda_1) = -W^{-2} B_1 \tilde{f}_1(-A).$$

Note that, we have

$$f_1(z) = -u_{d,r_+}^{-2}(z) v_{d,r_+}^{-2}(z) u_{c,r_-}^{-2}(z) v_{c,r_-}^{-2}(z) b_1(z) \tilde{f}_1(z). \tag{46}$$

Proposition 6.5. Assume that (H_1), (H_2), (H_3), (H_4) hold and $k_+k_- > 0$. Thus

- if $r_+ > 0$ and $r_- > 0$,
- if $r_+ < 0$ and $r_- < 0$, such that (39) holds,
- if $r_+ > 0$ and $r_- < 0$, such that (40) holds,
- if $r_+ < 0$ and $r_- > 0$, such that (41) holds,

then, $F_1 \in \mathcal{L}(X)$, given by (44), is invertible with bounded inverse.

Proof. For a given $\theta \in (0, \pi)$, we have $f_1, \tilde{f}_1 \in H(S_\theta)$. Moreover, for $z \in \mathbb{C} \setminus (-\infty, r]$, since

$$k_j b_1^{-1}(z) \left(-\sqrt{z + r_+}\right)^{l_j} \left(-\sqrt{z + r_-}\right)^{m_j} \left(-\sqrt{z}\right)^{n_j}, \quad \text{for all } j \in J,$$

are polynomial functions, we deduce that $1 - \tilde{f}_1 \in \mathcal{E}_\infty(S_\theta)$.

From (37), Proposition 6.3 and Remark 6.4, we know that $f_1 < 0$ and $b_2 > 0$ on $(r, +\infty)$. Then, since $u_{d,r_+}, u_{c,r_-}, v_{d,r_+}, v_{c,r_-} > 0$ on $(r, +\infty)$ and due to (45) and (46), we deduce that $\tilde{f}_1 < 0$ on $(r, +\infty)$.

Therefore, noting $\tilde{f_{r,1}}(z) = \tilde{f}_1(z + r)$ and applying Lemma 5.6 with $G = \tilde{f}_1$ and operator $P = -A - rI$, thus we deduce that operator $F_1 = \tilde{f_{r,1}}(-A - rI) = \tilde{f}_1(-A)$ is invertible with bounded inverse. $\square$

This result finally leads us to state the following main result of this section.

Proposition 6.6. Assume that (H_1), (H_2), (H_3), (H_4) hold and $k_+k_- > 0$. Thus

- if $r_+ > 0$ and $r_- > 0$,
- if $r_+ < 0$ and $r_- < 0$, such that (39) holds,
- if $r_+ > 0$ and $r_- < 0$, such that (40) holds,
- if $r_+ < 0$ and $r_- > 0$, such that (41) holds,

then $\det(\Lambda_1)$ is invertible with bounded inverse.

Proof. From (43), Proposition 6.3 and Proposition 6.5, it follows that $\det(\Lambda_1) = -W^{-2}B_1F_1$, is invertible with bounded inverse. □

6.2.2 Second case

Let $r_+ \in \mathbb{R}\setminus\{0\}$ and $r_- = 0$. In the same way than previously, using functional calculus, we prove that the determinant of system (25), given by (34), is invertible with bounded inverse. Due to Lemma 6.2, and the definition of D_4, we obtain:

$$D_3^+ = g_3^+(-A), \quad D_3^- = g_3^-(-A) \quad \text{and} \quad D_4 = g_4(-A),$$

where, for $z \in \mathbb{C}\setminus\mathbb{R}_-$, we have set

$$\begin{cases} g_3^+(z) &= 4k_+^2(\sqrt{z+r_+}+\sqrt{z})^2 u_{d,r_+}^{-2}(z) v_{d,r_+}^{-2}(z) g_{d,r_+}(z) \\ g_3^-(z) &= 16k_-^2\, z\, u_{c,0}^{-2}(z) v_{c,0}^{-2}(z) g_{c,0}(z) \\ g_4(z) &= -2\sqrt{z}\left(k_+ f_{d,r_+,3}(z) k_- f_{c,0,2}(z) + k_+ f_{d,r_+,2}(z) k_- f_{c,0,3}(z)\right) \\ & \quad +2z k_+ f_{d,r_+,1}(z) k_- f_{c,0,1}(z), \end{cases}$$

with $u_{\delta,r}$, $v_{\delta,r}$, $g_{\delta,r}$ and $f_{\delta,r,i}$ the complex functions defined in section 5.2. Thus

$$\det(\Lambda_2) = D_3^+ + D_3^- + D_4 = f_2(-A), \tag{47}$$

with $f_2 = g_3^+ + g_3^- + g_4$. Note that, for some $\theta \in (0,\pi)$, we have $f_2 \in H(S_\theta)$ and due to Remark 5.8 and Lemma 5.9, for $x > \max(-r_+, 0)$, we have

$$f_2(x) = g_3^+(x) + g_3^-(x) + g_4(x), \quad \text{where} \quad g_3^+ < 0 \quad \text{and} \quad g_3^-, g_4 > 0. \tag{48}$$

Lemma 6.7. Let $k_+k_- > 0$. Then

- if $r_+ > 0$ such that
$$r_+ \geqslant \frac{\left(\sqrt{t+1}+\sqrt{t}\right)^2}{t^2}\frac{k_+^2}{4k_-^2}, \quad \text{for } t > 0 \text{ fixed}. \tag{49}$$
for all $x \geqslant tr_+$, we have $f_2(x) > 0$.
- if $r_+ < 0$ such that
$$r_+ \leqslant -\frac{27k_+^2}{64k_-^2}, \tag{50}$$
for all $x \geqslant -r_+$, we have $f_2(x) > 0$.

Proof. From (48), we deduce

$$f_2(x) \geqslant g_3^+(x) + g_4(x) \geqslant g_3^+(x) + 2k_+k_-\, x\, f_{d,r_+,1}(x) f_{c,0,1}(x).$$

Let $r = \max(-r_+, 0)$. For $x \in (r, +\infty)$, setting $y = x - r > 0$ and noting

$$h_1(y) = g_3^+(y+r) + 2k_+k_-\,(y+r)\, f_{d,r_+,1}(y+r) f_{c,0,1}(y+r),$$

it follows

$$\begin{aligned} h_1(y) &= 4k_+^2 \frac{(\sqrt{y+r+r_+}+\sqrt{y+r})^2}{u_{d,r_+}^2(y+r) v_{d,r_+}^2(y+r)} g_{d,r_+}(y+r) \\ &\quad + 2k_+k_-\,(y+r)\, f_{d,r_+,1}(y+r) f_{c,0,1}(y+r). \end{aligned}$$

Since we have

$$0 > g_{d,r_+}(y+r) \geqslant -\sqrt{y+r+r_+}\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)^2,$$

then

$$h_1(y) \geqslant 4 \frac{(\sqrt{y+r+r_+}+\sqrt{y+r})\sqrt{y+r+r_+}}{u_{d,r_+}^2(y+r) v_{d,r_+}^2(y+r) u_{c,0}(y+r) v_{c,0}(y+r)} h_2(y),$$

where

$$\begin{aligned} h_2(y) &= -k_+^2(\sqrt{y+r+r_+}+\sqrt{y+r})\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)^2 u_{c,0}(y+r) v_{c,0}(y+r) \\ &\quad + k_+k_-\,(y+r)\left(1 - e^{-2c\sqrt{y+r}}\right)^2 v_{d,r_+}(y+r)\left(1 + e^{-d\sqrt{y+r}}\right)\left(1 + e^{-d\sqrt{y+r+r_+}}\right) \\ &\quad + k_+k_-\,(y+r)\left(1 - e^{-2c\sqrt{y+r}}\right)^2 u_{d,r_+}(y+r)\left(1 - e^{-d\sqrt{y+r}}\right)\left(1 - e^{-d\sqrt{y+r+r_+}}\right) \\ &\geqslant -k_+^2(\sqrt{y+r+r_+}+\sqrt{y+r})\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)^2\left(1 - e^{-2c\sqrt{y+r}}\right)^2 \\ &\quad + k_+k_-\,(y+r)\left(1 - e^{-2c\sqrt{y+r}}\right)^2 h_3(y), \end{aligned}$$

with

$$\begin{aligned} h_3(y) &= v_{d,r_+}(y+r)\left(1 + e^{-d\sqrt{y+r}}\right)\left(1 + e^{-d\sqrt{y+r+r_+}}\right) \\ &\quad + u_{d,r_+}(y+r)\left(1 - e^{-d\sqrt{y+r}}\right)\left(1 - e^{-d\sqrt{y+r+r_+}}\right) \end{aligned}$$

and

$$\begin{aligned} h_3(y) &= 2v_{d,r_+}(y+r)\left(1 + e^{-d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right) + 2u_{d,r_+}(y+r)\left(e^{-d\sqrt{y+r}} + e^{-d\sqrt{y+r+r_+}}\right) \\ &= 2\left(1 - e^{-d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)\left(1 + e^{-d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right) \\ &\quad + 2\frac{(\sqrt{y+r+r_+}+\sqrt{y+r})^2}{r_+}\left(e^{-d\sqrt{y+r}} - e^{-d\sqrt{y+r+r_+}}\right)\left(e^{-d\sqrt{y+r}} + e^{-d\sqrt{y+r+r_+}}\right) \\ &= 2\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right) \\ &\quad + 2\frac{(\sqrt{y+r+r_+}+\sqrt{y+r})^2}{r_+}\left(e^{-2d\sqrt{y+r}} - e^{-2d\sqrt{y+r+r_+}}\right). \end{aligned}$$

Moreover, for all $y > 0$, since we have

$$\frac{e^{-2d\sqrt{y+r}} - e^{-2d\sqrt{y+r+r_+}}}{r_+} > 0, \quad \text{for } r_+ \in \mathbb{R}\setminus\{0\},$$

we deduce that

$$h_3(y) > 2\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right),$$

and

$$\begin{aligned} h_2(y) &> -k_+^2(\sqrt{y+r+r_+}+\sqrt{y+r})\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)^2\left(1 - e^{-2c\sqrt{y+r}}\right)^2 \\ &\quad +2k_+k_-\,(y+r)\left(1 - e^{-2c\sqrt{y+r}}\right)^2\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right) \\ &> -k_+^2(\sqrt{y+r+r_+}+\sqrt{y+r})\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)\left(1 - e^{-2c\sqrt{y+r}}\right)^2 \\ &\quad +2k_+k_-\,(y+r)\left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)\left(1 - e^{-2c\sqrt{y+r}}\right)^2 \\ &> \left(1 - e^{-2d(\sqrt{y+r+r_+}+\sqrt{y+r})}\right)\left(1 - e^{-2c\sqrt{y+r}}\right)^2 h_4(y), \end{aligned}$$

where

$$h_4(y) = 2k_+k_-\,(y+r) - k_+^2(\sqrt{y+r+r_+}+\sqrt{y+r}).$$

Thus

$$h_4(y) \geqslant 0 \Longleftrightarrow \frac{y+r}{\sqrt{y+r+r_+}+\sqrt{y+r}} \geqslant \frac{k_+^2}{2k_+k_-}. \tag{51}$$

We set

$$h_5(y) = \frac{y+r}{\sqrt{y+r+r_+}+\sqrt{y+r}}, \tag{52}$$

hence

$$h_5'(y) = \left(\frac{1}{\sqrt{y+r+r_+}+\sqrt{y+r}}\right)\left(1 - \frac{1}{2}\sqrt{\frac{y+r}{y+r+r_+}}\right).$$

1. If $r_+ < 0$, then $r = -r_+$ and

$$\frac{y+r}{y+r+r_+} = \frac{y+r}{y},$$

moreover

$$h_5'(y) \geqslant 0 \Longleftrightarrow 1 - \frac{1}{2}\sqrt{\frac{y+r}{y}} \geqslant 0 \Longleftrightarrow 4 \geqslant \frac{y+r}{y} \Longleftrightarrow y \geqslant \frac{r}{3}.$$

Thus, we have

$$h_5(y) \geqslant h_5\left(\frac{r}{3}\right) = \frac{\frac{4r}{3}}{\sqrt{\frac{r}{3}}+2\sqrt{\frac{r}{3}}} = \frac{4}{3}\sqrt{\frac{r}{3}} > 0.$$

This yields that for all $y \geqslant 0$, we have

$$h_5(y) \geqslant \frac{4}{3}\sqrt{\frac{r}{3}} > 0.$$

Therefore, from (51) and (52), we deduce that

$$h_4(y) \geqslant 0 \Longleftrightarrow h_5(y) \geqslant \frac{k_+^2}{2k_+k_-} \Longleftrightarrow \frac{4}{3}\sqrt{\frac{r}{3}} \geqslant \frac{k_+}{2k_-},$$

hence, since $r = -r_+ > 0$, we obtain

$$\frac{4}{3}\sqrt{\frac{r}{3}} \geqslant \frac{k_+}{2k_-} \Longleftrightarrow \sqrt{\frac{r}{3}} \geqslant \frac{3k_+}{8k_-} \Longleftrightarrow \frac{r}{3} \geqslant \frac{9k_+^2}{64k_-^2} \Longleftrightarrow -r_+ \geqslant \frac{27k_+^2}{64k_-^2}.$$

2. If $r_+ > 0$, then $r = 0$ and
$$\frac{y+r}{y+r+r_+} = \frac{y}{y+r_+} < 1,$$
hence $h_5' > 0$ and h_5 is an increasing function. Thus, from (52), since $r = 0$, it follows that
$$h_5(y) = \frac{y}{\sqrt{y+r_+} + \sqrt{y}},$$
then
$$h_5(y) \geqslant \frac{k_+^2}{2k_+k_-} \Longleftrightarrow \frac{y}{\sqrt{y+r_+}+\sqrt{y}} \geqslant \frac{k_+}{2k_-}.$$
Moreover, for $t > 0$ fixed, we have
$$h_5(tr_+) \geqslant \frac{k_+}{2k_-} \Longleftrightarrow \frac{tr_+}{\sqrt{(t+1)r_+}+\sqrt{tr_+}} \geqslant \frac{k_+}{2k_-} \Longleftrightarrow \frac{t}{\sqrt{t+1}+\sqrt{t}}\sqrt{r_+} \geqslant \frac{k_+}{2k_-},$$
hence
$$\sqrt{r_+} \geqslant \frac{\sqrt{t+1}+\sqrt{t}}{t}\frac{k_+}{2k_-} \Longleftrightarrow r_+ \geqslant \frac{\left(\sqrt{t+1}+\sqrt{t}\right)^2}{t^2}\frac{k_+^2}{4k_-^2}.$$

Finally, if $r_+ > 0$ such that (49) holds, then since $y = x$, for all $x \geqslant tr_+$, we have $h_2(x) > 0$, $h_1(x) > 0$ and $f_2(x) > 0$. Moreover, $r_+ < 0$ such that (50) holds, then for all $y > 0$, we have $h_2(y) > 0$, $h_1(y) > 0$ and since $y = x + r_+$, for all $x > -r_+$, it follows that $f_2(x) > 0$. □

Therefore, as in the first case, since we have $U_\pm \sim I$ and $V_\pm \sim I$, then by setting $W = U_-U_+V_-V_+ \sim I$, we deduce that
$$\begin{cases} WP_1^+ \sim 2k_+(L_+ + M), & WQ_1^- \sim 2k_-I \\ WP_2^+ \sim 2k_+(L_+ + M), & WQ_2^- \sim 2k_-I \\ WP_3^+ \sim 2k_+(L_+ + M)L_+, & WQ_3^- \sim 2k_-I. \end{cases}$$
Thus
$$\begin{aligned} W^2\det(\Lambda_2) &= \left(WP_2^+WP_3^+ - M\left(WP_1^+\right)^2\right) + 4M^2\left(WQ_2^-WQ_3^- - M\left(WQ_1^-\right)^2\right) \\ &\quad +2M\left(WP_3^+WQ_2^- + WP_2^+WQ_3^- + MWP_1^+WQ_1^-\right) \\ &\sim 4k_+^2(L_+ + M)^2(L_+ - M) + 16k_-^2M^2(I - M) \\ &\quad +8k_+k_-(L_+ + M)M(L_+ + M + I). \end{aligned}$$
From (7), we have
$$\begin{aligned} W^2\det(\Lambda_2) &\sim 4k_+^2r_+(L_+ + M) + 16k_-^2M^2(I - M) \\ &\quad +8k_+k_-(L_+ + M)M(L_+ + M + I) \\ &\sim 4k_+l_+(L_+ + M) + 16k_-^2M^2(I - M) \\ &\quad +8k_+k_-(L_+ + M)M(L_+ + M + I) \end{aligned}$$
Hence, we note
$$B_3 = 4k_+l_+(L_+ + M) + 16k_-^2M^2(I - M) + 8k_+k_-(L_+ + M)M(L_+ + M + I).$$

Thus, we obtain

$$\det(\Lambda_2) = W^{-2}\left(B_3 + \sum_{j\in J} k_j L_+^{l_j} M^{m_j} e^{\alpha_j L_+} e^{\beta_j M}\right), \tag{53}$$

where J is a finite set and for any $j \in J$:

$$k_j \in \mathbb{R};\ l_j, m_j \in \mathbb{N},\ \alpha_j, \beta_j \in \mathbb{R}_+ \quad \text{with} \quad \alpha_j + \beta_j \neq 0.$$

Proposition 6.8. Assume that (H_1), (H_2), (H_3) hold and $k_+k_- > 0$. If $\dfrac{k_-}{k_+} \leqslant 2$, then

$$0 \in \rho\left(8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3\right).$$

Proof. Since $k_+k_- > 0$, we have $\dfrac{k_-}{k_+} > 0$ and

$$8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3 = 8k_+k_- M\left[L_+^2 + 2L_+M + M^2 - 2\frac{k_-}{k_+}M^2\right].$$

From Remark 4.1, statement 5 and [31], Corollary 3, p. 444, we deduce that

$$L_+^2,\ 2L_+M,\ M^2 \in \mathrm{BIP}(X, \theta_A).$$

Thus, if $\dfrac{k_-}{k_+} \leqslant 1$, then

$$L_+^2 + 2L_+M + M^2 - 2\frac{k_-}{k_+}M^2 = L_+^2 - \frac{k_-}{k_+}M^2 + 2L_+M + \left(1 - \frac{k_-}{k_+}\right)M^2.$$

Moreover, for all $\psi \in D(M^2) = D(A)$, due to (6), we have

$$\left(L_+^2 - \frac{k_-}{k_+}M^2\right)\psi = \left[-\left(1 - \frac{k_-}{k_+}\right)A + r_+ I\right]\psi,$$

and from [31], Theorem 3, p. 437 and [1], Theorem 2.3, p. 69, assumptions (H_2) and (H_3) imply that

$$-\left(1 - \frac{k_-}{k_+}\right)A + r_+ I \in \mathrm{BIP}(X, \theta_A),$$

and

$$L_+^2 - \frac{k_-}{k_+}M^2 + 2L_+M + \left(1 - \frac{k_-}{k_+}\right)M^2 \in \mathrm{BIP}(X, \theta_A + \varepsilon),$$

for any $\varepsilon \in (0, \pi - \theta_A)$. Moreover, since $0 \in \rho(L_+M)$, we deduce from [31], remark at the end of p. 445, that $0 \in \rho\left(L_+^2 - \frac{k_-}{k_+}M^2 + 2L_+M + \left(1 - \frac{k_-}{k_+}\right)M^2\right)$. Therefore, since $0 \in \rho(M)$ and $k_+k_- > 0$, it follows that

$$0 \in \rho\left(8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3\right).$$

In the same way, if $1 < \dfrac{k_-}{k_+} \leqslant 2$, then

$$L_+^2 + 2L_+M + M^2 - 2\frac{k_-}{k_+}M^2 = L_+^2 - M^2 + 2L_+M - 2\left(\frac{k_-}{k_+} - 1\right)M^2 + M^2 - M^2,$$

hence, for all $\psi \in D(M^2) = D(A)$, from (6), we obtain

$$\left(L_+^2 + 2L_+M + M^2 - 2\frac{k_-}{k_+}M^2\right)\psi = r_+\psi + 2M\left(L_+ - \left(\frac{k_-}{k_+} - 1\right)M\right)\psi. \quad (54)$$

Moreover, we have

$$\left(L_+ - \left(\frac{k_-}{k_+} - 1\right)M\right)\psi = \left(L_+ + \left(\frac{k_-}{k_+} - 1\right)M\right)^{-1}\left(L_+^2 - \left(\frac{k_-}{k_+} - 1\right)^2 M^2\right)\psi,$$

and from [31], Theorem 3, p. 437 and [1], Theorem 2.3, p. 69, assumptions (H_2) and (H_3) imply that

$$\left(L_+^2 - \left(\frac{k_-}{k_+} - 1\right)^2 M^2\right) = -\left(2 - \frac{k_-}{k_+}\right)A + r_+I \in \mathrm{BIP}(X, \theta_A). \quad (55)$$

Finally, from (H_2), (H_3), (54), (55) and [31], Theorem 3, p. 437, we deduce that

$$r_+\psi + 2M\left(L_+ - \left(\frac{k_-}{k_+} - 1\right)M\right) \in \mathrm{BIP}(X, \theta_A),$$

and

$$0 \in \rho\left(r_+\psi + 2M\left(L_+ - \left(\frac{k_-}{k_+} - 1\right)M\right)\right).$$

Therefore, since $0 \in \rho(M)$ and $k_+k_- > 0$, it follows that

$$0 \in \rho\left(8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3\right).$$

□

We set

$$\begin{aligned} B_4 &= I + 4k_+l_+(L_+ + M)\left(8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3\right)^{-1} \\ &\quad + 16k_-^2 M^2\left(8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3\right)^{-1} \\ &\quad + 8k_+k_-(L_+ + M)M\left(8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3\right)^{-1}, \end{aligned} \quad (56)$$

thus, we have

$$B_3 = \left(8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3\right)B_4.$$

Moreover, from (53) and noting $B_5 = 8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3$, we have

$$\det(\Lambda_2) = W^{-2}B_5F_2, \quad (57)$$

where

$$F_2 = B_4 + \sum_{j\in J} k_j B_5^{-1} L_+^{l_j} M^{m_j} e^{\alpha_j L_+} e^{\beta_j M}. \quad (58)$$

Now, for $z \in \mathbb{C} \setminus [\max(-r_+, 0), +\infty)$, we set

$$\tilde{f}_2(z) = b_4(z) + \sum_{j\in J} k_j b_5(z)^{-1} \sqrt{z + r_+}^{l_j} \sqrt{z}^{m_j} e^{-\alpha_j\sqrt{z+r_+}} e^{-\beta_j\sqrt{z}},$$

where $b_3(z) = b_4(z)b_5(z)$, with

$$\begin{aligned} b_4(z) &= 1 + 4k_+l_+(\sqrt{z+r_+} + \sqrt{z}) \left(8k_+k_-(\sqrt{z+r_+} + \sqrt{z})^2\sqrt{z} - 16k_-^2\sqrt{z}^3\right)^{-1} \\ &\quad +16k_-^2 z \left(-8k_+k_-(\sqrt{z+r_+} + \sqrt{z})^2\sqrt{z} + 16k_-^2\sqrt{z}^3\right)^{-1} \\ &\quad +8k_+k_-(\sqrt{z+r_+} + \sqrt{z})\sqrt{z} \left(8k_+k_-(\sqrt{z+r_+} + \sqrt{z})^2\sqrt{z} - 16k_-^2\sqrt{z}^3\right)^{-1}, \end{aligned}$$

and

$$b_5(z) = -8k_+k_-(\sqrt{z+r_+} + \sqrt{z})^2\sqrt{z} + 16k_-^2\sqrt{z}^3.$$

Then $\tilde{f}_2(-A) = F_2$, $b_3(-A) = B_3$, $b_4(-A) = B_4$ and $b_5(-A) = B_5$. Thus, from (47) and (57), we deduce that

$$f_2(z) = u_{d,r_+}^{-2}(z)v_{d,r_+}^{-2}(z)u_{c,0}^{-2}(z)v_{c,0}^{-2}(z)b_5(z)\tilde{f}_2(z). \tag{59}$$

Proposition 6.9. Assume that (H_1), (H_2), (H_3), (H_4) hold and $k_+k_- > 0$ with $\dfrac{k_-}{k_+} \leqslant 2$. Thus

- if $r_+ > 0$ such that

$$r_+ \geqslant \frac{\left(\sqrt{t+1} + \sqrt{t}\right)^2}{t^2} \frac{k_+^2}{4k_-^2}, \quad \text{for } t \in \left(0, \frac{1}{r_+\|A^{-1}\|_{\mathcal{L}(X)}}\right) \text{ fixed}, \tag{60}$$

- if $r_+ < 0$ such that (50) holds,

then F_2, given by (58), is invertible with bounded inverse.

Proof. From Proposition 6.8 and (58), we deduce that F_2 is well defined.

- Assume that $r_+ > 0$ such that (49) holds. Then, from Lemma 6.7 and (59), it follows that f_2 does not vanish on $(tr_+, +\infty)$, for $t > 0$ fixed, which involves that u_{d,r_+}^{-2}, v_{d,r_+}^{-2}, $u_{c,0}^{-2}$, $v_{c,0}^{-2}$, b_5 and $\tilde{f}_2$ do not vanish on $(tr_+, +\infty)$, for $t > 0$ fixed. Moreover, due to (H_2), there exists $R = \frac{1}{\|A^{-1}\|_{\mathcal{L}(X)}} > 0$ such that $B(0,R) \subset \rho(A)$. Therefore, setting $\tilde{f}_{tr_+,2}(z) = \tilde{f}_2(z + tr_+)$, with $t \in \left(0, \frac{1}{r_+\|A^{-1}\|_{\mathcal{L}(X)}}\right)$ fixed and applying Lemma 5.6 where we have set $G = \tilde{f}_{tr_+,2}$ and operator $P = -A - tr_+I \in \text{BIP}\,(X, \theta_A)$ (due to (H_2) and (H_3)), we deduce that operator $F_2 = \tilde{f}_{tr_+,2}(-A - tr_+I) = \tilde{f}_2(-A)$ is invertible with bounded inverse.
- Now, assume that $r_+ < 0$ such that (50) holds. Then $\tilde{f}_2$ does not vanish on $(-r_+, +\infty)$. Moreover, from (H_2) and (H_3), we have $-A + r_+I \in \text{BIP}\,(X, \theta_A)$. It follows that $F_2 = \tilde{f}_{-r_+,2}(-A + r_+I) = \tilde{f}_2(-A)$ is invertible with bounded inverse.

□

This result finally leads us to state the following main result of this section.

Proposition 6.10. Assume that (H_1), (H_2), (H_3), (H_4) hold and $k_+k_- > 0$ with $\dfrac{k_-}{k_+} \leqslant 2$. Thus

- if $r_+ > 0$ such that (60) holds,
- if $r_+ < 0$ such that (50) holds,

then $\det(\Lambda_2)$ is invertible with bounded inverse.

Proof. From (57), Proposition 6.8 and Proposition 6.9, we obtain that $\det(\Lambda_2) = W^{-2}B_5F_2$, is invertible with bounded inverse. □

6.3 Regularity

6.3.1 First case

Here, we consider $r_+, r_- \in \mathbb{R} \setminus \{0\}$. From Theorem 5.3, we have to prove that system (17) has a unique solution (ψ_1, ψ_2) satisfying (18). The existence and uniqueness of this solution is ensured by Proposition 6.6, so we have

$$\left\{\begin{array}{rcl} \psi_1 &=& \left(P_1^- - P_1^+\right)[\det(\Lambda_1)]^{-1} S_1 - \left(P_2^+ + P_2^-\right)[\det(\Lambda_1)]^{-1} S_2 \\ \\ \psi_2 &=& -\left(P_3^+ - P_3^-\right)[\det(\Lambda_1)]^{-1} S_1 + M\left(P_1^- - P_1^+\right)[\det(\Lambda_1)]^{-1} S_2. \end{array}\right. \tag{61}$$

Now, we have to study the regularity of $[\det(\Lambda_1)]^{-1}$. Since, in this case, the determinant $\det(\Lambda_1)$ is the same than the one in [19], we deduce, from [19], Lemma 5.3, p. 2958, that there exists $R_{\det(\Lambda_1)} \in \mathcal{L}(X)$ such that

$$R_{\det(\Lambda_1)}(X) \subset D(M), \quad [\det(\Lambda_1)]^{-1} = N^{-1} + N^{-1} R_{\det(\Lambda_1)},$$

where $N = 4k_+k_-(L_- + M)(L_+ + M)(L_+ + L_- + 2M)$. Then, the rest of the proof is similar to the one given in [19], section 5.3. Therefore, from (12) and (13), it follows that $S_1, S_2 \in (D(M), X)_{1+\frac{1}{p},p}$ and thus

$$[\det(\Lambda)]^{-1} S_1, [\det(\Lambda)]^{-1} S_2 \in (D(M), X)_{4+\frac{1}{p},p}. \tag{62}$$

Moreover, from (61), we have

$$\left\{\begin{array}{rcl} \psi_1 &=& -2\left(k_+(L_+ + M) - k_-(L_- + M)\right)[\det(\Lambda_1)]^{-1} S_1 \\ && +2\left(k_+(L_+ + M) - k_-(L_- + M)\right)[\det(\Lambda_1)]^{-1} S_2 + \tilde{S}_1 \\ \\ \psi_2 &=& -2\left(k_+(L_+ + M)L_+ + k_-(L_- + M)L_-\right)[\det(\Lambda_1)]^{-1} S_1 \\ && -2\left(k_+(L_+ + M) - k_-(L_- + M)\right)[\det(\Lambda_1)]^{-1} S_2 + \tilde{S}_2, \end{array}\right. \tag{63}$$

where $\tilde{S}_1$, $\tilde{S}_2 \in D(M^\infty)$. Finally, from (27), (62) and (63), we obtain

$$\left\{\begin{array}{l} \psi_1 \in (D(M), X)_{3+\frac{1}{p},p} = (D(A), X)_{1+\frac{1}{2p},p} \\ \psi_2 \in (D(M), X)_{2+\frac{1}{p},p} = (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \end{array}\right.$$

6.3.2 Second case

Here, we consider $r_+ \in \mathbb{R} \setminus \{0\}$ and $r_- = 0$. From Theorem 5.5, we have to prove that (25) has a unique solution (ψ_1, ψ_2) satisfying (18). The existence and uniqueness of this solution is ensured by Proposition 6.10, so we have

$$\left\{\begin{array}{rcl} \psi_1 &=& \left(2MQ_1^- - P_1^+\right)[\det(\Lambda_2)]^{-1} S_3 + \left(P_2^+ + 2MQ_2^-\right)[\det(\Lambda_2)]^{-1} S_4 \\ \\ \psi_2 &=& -\left(P_3^+ + 2MQ_3^-\right)[\det(\Lambda_2)]^{-1} S_3 + M\left(P_1^+ - 2MQ_1^-\right)[\det(\Lambda_2)]^{-1} S_4. \end{array}\right. \tag{64}$$

Now, we have to study the regularity of $[\det(\Lambda_2)]^{-1}$. From (9), (56), (57), (58) and [20], Lemma 5.1, p. 365, we deduce that there exists $R_{\det(\Lambda_2)} \in \mathcal{L}(X)$ such that

$$R_{\det(\Lambda_2)}(X) \subset D(M), \quad [\det(\Lambda_2)]^{-1} = B_5^{-1} + B_5^{-1} R_{\det(\Lambda_2)},$$

where we recall that $B_5 = 8k_+k_-(L_+ + M)^2 M - 16k_-^2 M^3$. Moreover, from (8), (27), (29) and (32), we have

$$\tilde{\varphi}_1^+, \tilde{\varphi}_2^-, \tilde{\varphi}_2^+, \tilde{\varphi}_3^+, \tilde{\varphi}_4^-, \tilde{\varphi}_4^+ \in (D(M), X)_{2+\frac{1}{p},p} \quad \text{and} \quad \tilde{\varphi}_1^-, \tilde{\varphi}_3^- \in (D(M), X)_{3+\frac{1}{p},p}.$$

Thus, from (21), (22), (23), Remark 5.2 and Remark 5.4, we deduce that

$$R_2 \in (D(M), X)_{\frac{1}{p},p} \quad \text{and} \quad S_3, S_4 \in (D(M), X)_{1+\frac{1}{p},p},$$

which implies that

$$[\det(\Lambda_2)]^{-1} S_3, \ [\det(\Lambda_2)]^{-1} S_4 \in (D(M), X)_{4+\frac{1}{p},p}. \tag{65}$$

Moreover, due to (10), (20), (64) and [20], Lemma 5.1, p. 365, we have

$$\left\{ \begin{array}{lll} \psi_1 & = & -2\left(k_+(L_+ + M) - 2k_- M\right) [\det(\Lambda_2)]^{-1} S_3 \\ & & +2\left(k_+(L_+ + M) + 2k_- M\right) [\det(\Lambda_2)]^{-1} S_4 + \tilde{S}_3 \\ \psi_2 & = & -2\left(k_+(L_+ + M)L_+ + 2k_- M\right) [\det(\Lambda_2)]^{-1} S_3 \\ & & +2M\left(k_+(L_+ + M) - 2k_- M\right) [\det(\Lambda_2)]^{-1} S_4 + \tilde{S}_4, \end{array} \right. \tag{66}$$

where $\tilde{S}_3$, $\tilde{S}_4 \in D(M^\infty)$. Finally, from (27), (65) and (66), we obtain

$$\left\{ \begin{array}{l} \psi_1 \in (D(M), X)_{3+\frac{1}{p},p} = (D(A), X)_{1+\frac{1}{2p},p} \\ \psi_2 \in (D(M), X)_{2+\frac{1}{p},p} = (D(A), X)_{1+\frac{1}{2}+\frac{1}{2p},p}. \end{array} \right.$$

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