Applications of computer algebra in the identifiability and diagnosability studies

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Outline

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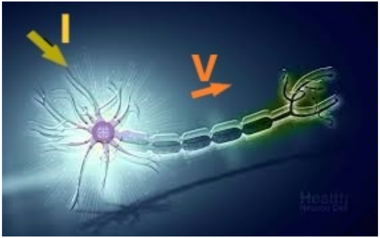
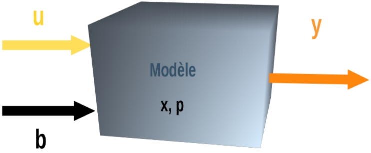
   - State of the problem
   - A short quiz
   - A first neuron model
   - Formalization of the identifiability definition
   - Towards the relative identifiability with a (little) more complex example
   - Conclusion
2. Fault diagnosability
   - State of the problem
   - Algebraic signature
   - Characterization of a single fault/Expected values of *ASig*
   - Conclusion
3. Tutorial
   - Relative identifiability
   - Diagnosability
4. Bibliography

Considered model

Given a *system* or a *process*, some quantities interact:



Neuron responding to a electrical signal



Schematic representation of the functioning of the neuron

- Electrical signal (I) $\leftrightarrow$ $u = I$ = *input*
- Membrane potential (V) $\leftrightarrow$ $y = V$ = *output*
- *Perturbations* $\leftrightarrow$ b
- Non measurable varying quantities $\leftrightarrow$ x
- Unknown constant values $\leftrightarrow$ p = vector of unknown parameters.

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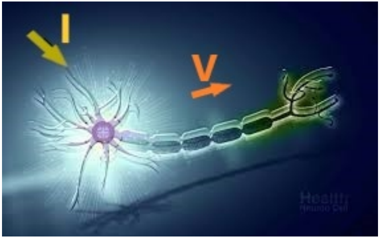

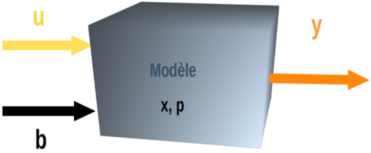

Neuron responding to a electrical signal

Schematic representation of the functioning of the neuron

$$\begin{cases} \dot{x}(t,p) = f(x(t,p),u(t),p), \\ y(t,p) = h(x(t,p),p). \end{cases} \quad (1)$$

- ✓ f, h: real functions, analytic on M (an open set of $\mathbb{R}^n$),
- ✓ $p \in \mathcal{U}_{\mathcal{P}}$: vector of parameters,
  $\mathcal{U}_{\mathcal{P}} \subset \mathbb{R}^r$: an a priori known set of admissible parameters.

Assume that the process can be modeled by

$$\begin{cases} \dot{x}(t,p) = f(x(t,p),u(t),p), \\ y(t,p) = h(x(t,p),p). \end{cases}$$

Two problems can be considered:

- ✓ The *forward problem*: given p, u, find x and y.
- ✓ The *inverse problem*: given y and u, estimate p.
  1. Identifiability problem

Question

From the output(s) of the system, is it possible to estimate uniquely the parameter vector p?
If the answer is YES, then the model is said identifiable.

  2. Identification problem

Assume that the process can be modeled by

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Two problems can be considered:

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- ✓ The *inverse problem*: given y and u, estimate p.

A property of lots of inverse problems: *ill-posedness*.

Well-posedness is the sense of Hadamard

A problem is said well-posed in the sense of Hadamard if it satisfies the following properties:

1. Existence: For all (suitable) data, there exists a solution of the problem (in an appropriate sense)
2. Unicity : for all available data, the solution is unique
3. Stability : the solution depends continuously of the data.

Example: The differentiation and integration are two inverse problems of each other.

What about the neuron?

- ✓ Microscopic worm Caenorhabditis elegans ($\approx 1mm$ de long) has .... neurons
- ✓ The insects have approximatively ..... of neurons
- ✓ Modern man has .... of neurons in its best form
- ✓ Every day we loose approximatively ..... neurons, which is the equivalent of ......
- ✓ At 80 years old, the brain is only ..... percent of what it was around 20 years
- ✓ Nerve information passes from neurons to neurons, up to ......

What about the neuron?

- ✓ Microscopic worm Caenorhabditis elegans ($\approx 1mm$ de long) has 302 neurons
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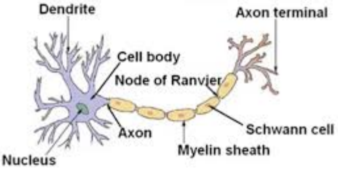
Neuron

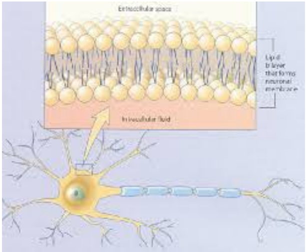
A neuron is a nerve cell, that is an electrically excitable cell that receives, processes, and transmits information through electrical and chemical signals.



Membrane

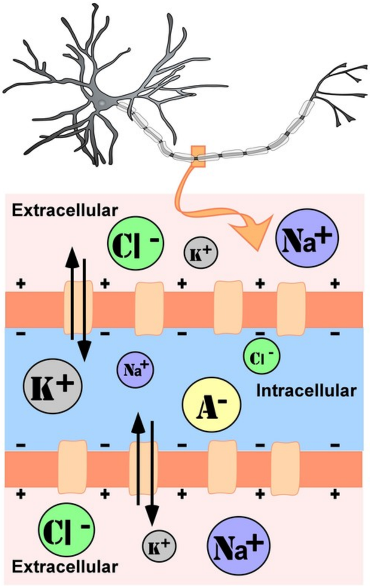
A lipid membrane

A membrane is composed of a lipid bilayer which separates the intracellular milieu and the extracellular milieu.



The main ions in a neuron are:

- Sodium (Na^{+})
- Potassium (K^{+})
- Calcium (Ca^{2+})
- Chlorure (Cl^{-})



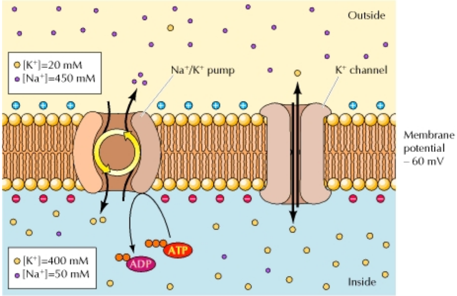
- Unequally distribution of ions

  on both sides of the membrane

- Specific channels for each ion

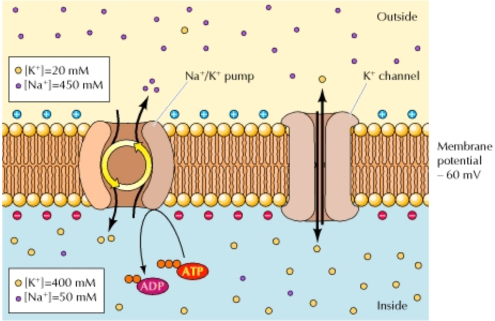
- Channels can be:
  - ✓ open or close
  - ✓ active or inactive.

However there is electroneutrality!



Some mechanisms permit to regulate ionic concentrations and to maintain them constant. Two types of transport:

- ✓ passive transport:
  - the concentration gradient: ions go from the most concentrated milieu to the least concentrated milieu
    extracellular → intracellular: Cl^- , Na^+ , Ca^{2+}
    intracellular → extracellular: K^+
  - electrical gradient: the membrane is electrically charged: negatively inside, positively outside
    extracellular → intracellular: K^+, Na^+ et Ca^{2+} are attracted inside the neuron
    intracellular → extracellular: Cl^-
- ✓ active transport (NA/K pump) requiring energy.



Potential differences:

- ✓ the potential difference of the membrane:
  $$V_m = \underbrace{V_i}_{\text{intracellular potential}} - \underbrace{V_e}_{\text{extracellular potential}}$$
- ✓ the potential difference due to the passage of an ion:
  $$V_m - \underbrace{E_{ion}}_{\text{equilibrium potential of an ion}} .$$

The current due to the passage of an ion in a channel:

$$\underbrace{V_m - E_{ion}}_{mV} = \underbrace{r}_{\text{resistance of channel}} \times \underbrace{I_{ion}}_{\text{ionic current (pA)}} \Leftrightarrow I_{ion} = \underbrace{G}_{\text{conductance (siemens } S)} (V_m - E_{ion}).$$

Modeling of a simple ion channel with one activation (m):

$$I = G\,(V - E) \text{ where } G = g\,m$$

where

- $V\,(mV)$: voltage
- $g\,(nS)$: maximal conductance
- $E\,(mV)$: equilibrium potential
- m is the probability of a canal to be open

Modeling of a simple ion channel with one activation (m):

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- $V\,(mV)$: voltage
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- $E\,(mV)$: equilibrium potential
- m is the probability of a canal to be open

The equation describing the activation of the gates to the answer of the potential of membrane is

$$\frac{dm}{dt} = \frac{m_\infty(V) - m}{\tau(V)}$$

where

- $m_\infty(V)$: the equilibrium value of m,
- $\tau(V)$: times at which the equilibrium is attained.

Question

Is it possible to determine in a unique way m_∞, τ, g from the measurement of the current?

What can we measure?

The *voltage-clamp protocol*

- characterizes the activation or inactivation properties of the ionic canal
- necessitates to treat the membrane of the neuron (tetrodoxine)
- consists in holding the voltage ($= V$) piecewise constant
  $\hookrightarrow$ after a while, m_∞, τ can be considered as constant and we have

$$\frac{dm}{dt} = \frac{m_\infty - m}{\tau}$$

.

Assumptions

- ✓ V = constant input
- ✓ I = output (= y)
- ✓ m = state variable (= x)

A first example: The equation of one ion channel with one activation variable:

$$\begin{cases} I &= g\,m\,(V-E) \\ \dfrac{dm}{dt} &= \dfrac{m_\infty - m}{\tau} \end{cases} \underset{\substack{\text{u:=V-E=cst,}\\ \text{y:=I}}}{\Longleftrightarrow} \begin{cases} y &= g\,m\,u \\ \dfrac{dm}{dt} &= \dfrac{m_\infty - m}{\tau} \end{cases}$$

Controlled models ($u \neq 0$) WITHOUT initial condition; $(\bar{x},\ \bar{y})$ = unique set of solutions

- The model is **globally identifiable** if there exists an input u such that, for all $p \in \mathcal{U}_p$, one gets

$$\left.\begin{array}{c} \bar{y}(t,p) \neq \emptyset, \\ \bar{y}(t,p) \cap \bar{y}(t,\bar{p}) \neq \emptyset,\ \forall\, t \geq 0,\ \bar{p} \in \mathcal{U}_p \end{array}\right\} \quad \Rightarrow \quad p = \bar{p}. \tag{2}$$

- The model is **locally identifiable** if it is globally identifiable in an open neighborhood $v(p) \subset \mathcal{U}_p$ of p.

Proposition

If V is a constant input and I is an output of the model then the model is not identifiable, in particular with respect to τ and m_∞.

Proof.

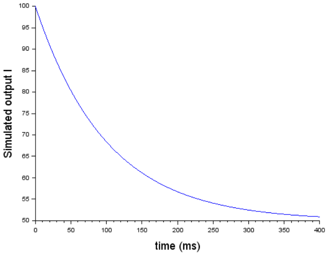
...

□

A first example: The equation of one ion channel with one activation variable:

$$\begin{cases} I & = g\, m\,(V - E) \\ \dfrac{dm}{dt} & = \dfrac{m_\infty - m}{\tau} \end{cases} \underset{\substack{u:=V-E=cst,\\ y:=I}}{\Longleftrightarrow} \begin{cases} y & = g\, m\, u \\ \dfrac{dm}{dt} & = \dfrac{m_\infty - m}{\tau} \end{cases}$$

Concretely



$\tau = 100ms$, $u = 20mV$,
$(m(0), m_\infty, g) = (1, 0.5, 5)$ and
$(m(0), m_\infty, g) = (2, 1, 2.5)$.

The model can produce exactly the same output for different parameter/initial condition values!

A first example: The equation of one ion channel with one activation variable:

$$\begin{cases} I & = g\, m\,(V - E) \\ \dfrac{dm}{dt} & = \dfrac{m_\infty - m}{\tau} \\ m(0) & = m_0 \end{cases} \underset{\text{u:=V-E=cst, y:=I}}{\Longleftrightarrow} \begin{cases} y & = g\, m\, u \\ m & = m_\infty + (m_0 - m_\infty)\, e^{-\frac{t}{\tau}} \end{cases}$$

Controlled model ($u \neq 0$) WITH initial conditions; (x, y) unique solution

- The model is **globally identifiable** if there exists an input u such that, for all $p, \bar{p} \in \mathcal{U}_p$, there exists $t_1 > 0$ such that if for all $t \in [0, t_1]$, the equalities $y(t, p) = y(t, \bar{p})$ implies that $p = \bar{p}$.
- The model is **locally identifiable** if it is globally in an open neighborhood $v(p) \subset \mathcal{U}_p$ of p.

Proposition

If V is a constant input and I is an output of the model then the model is identifiable, in particular τ and m_∞.

Proof.

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Summary

- ✓ Identifiability: based on specific relations called Input-Output (IO) polynomials.
- ✓ The Rosenfeld-Groebner algorithm permits to obtain them. They take the form

$$P(y, u, p) = m_0(y, u) + \sum_{k=1}^{q} \gamma_k(p) m_k(y, u) = 0.$$

- ✓ If $(m_k(y, u))_{k=1,\ldots,q}$ are linearly independent, the model is globally identifiable at p if for all $\bar{p} \in \mathcal{U}_p$

$$\forall k = 1, \ldots, q,\ \gamma_k(\bar{p}) = \gamma_k(p) \Rightarrow p = \bar{p}. \quad (3)$$

- ✓ If $\phi(p) = (\gamma_k(p))_{k=1,\ldots,q}$, (3) consists in verifying that ϕ is injective.
- ✓ Initial conditions can be introduced with algebraic relations.

A (little) more complex example....

$$\left\{\begin{array}{l} I = g m h (V - E), \\ \dfrac{dm}{dt} = \dfrac{m_\infty - m}{\tau_m}, \\ \dfrac{dh}{dt} = \dfrac{h_\infty - h}{\tau_h}, \end{array}\right. \quad \underset{\substack{x_1 = m,\, x_2 = h,\, u = V - E \\ p_1 = \frac{1}{\tau_m},\, p_2 = m_\infty, \\ p_3 = \frac{1}{\tau_h},\, p_4 = h_\infty,\, p_5 = g,}}{\Longleftrightarrow} \quad \left\{\begin{array}{l} y = u\, p_5\, x_1\, x_2, \\ \dot{x}_1 = p_1\,(p_2 - x_1), \\ \dot{x}_2 = p_3\,(p_4 - x_2). \end{array}\right.$$

Non identifiable model!

A (little) more complex example....

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Questions :

- ✗ Key parameters permitting to obtain the identifiability of one or some non measurable parameters and eventually the identifiability of the model?
- ✗ Roles of the constraints?
- ✗ Natural integration of the constraints or the initial conditions in the identifiability study?

$$\left\{\begin{array}{l} I = g m h (V - E), \\ \dfrac{dm}{dt} = \dfrac{m_\infty - m}{\tau_m}, \\ \dfrac{dh}{dt} = \dfrac{h_\infty - h}{\tau_h}, \end{array}\right. \iff \left\{\begin{array}{l} y = u\, p_5\, x_1\, x_2, \\ \dot{x}_1 = p_1\,(p_2 - x_1), \\ \dot{x}_2 = p_3\,(p_4 - x_2). \end{array}\right.$$

Obtain from ϕ and a set of algebraic constraints a **decision tree**!

$$
\left\{
\begin{array}{l}
I = gmh(V-E),\\
\dfrac{dm}{dt} = \dfrac{m_\infty - m}{\tau_m},\\
\dfrac{dh}{dt} = \dfrac{h_\infty - h}{\tau_h},
\end{array}
\right.
\iff
\left\{
\begin{array}{l}
y = u\, p_5\, x_1\, x_2,\\
\dot{x}_1 = p_1\,(p_2 - x_1),\\
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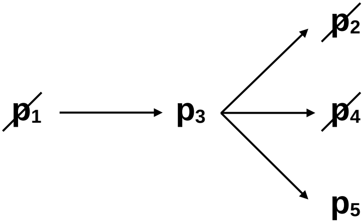
\end{array}
\right.
$$

First step: Redefine the identifiability: Relative identifiability
For example:



- p_1 is not identifiable;
- p_3 is relatively identifiable with respect to the set $\{p_1\}$;
- p_5 relatively identifiable with respect to the set $\{p_1, p_3\}$;
- p_2 and p_4 are not relatively identifiable with respect to the set $\{p_1, p_3\}$.

$$\left\{\begin{array}{l} I = gmh(V-E), \\ \dfrac{dm}{dt} = \dfrac{m_\infty - m}{\tau_m}, \\ \dfrac{dh}{dt} = \dfrac{h_\infty - h}{\tau_h}, \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} y = u\, p_5\, x_1\, x_2, \\ \dot{x}_1 = p_1\,(p_2 - x_1), \\ \dot{x}_2 = p_3\,(p_4 - x_2). \end{array}\right.$$

Second step: definition of a semi-algebraic set from

- $\phi(p) = (\gamma_k(p))_{k=1,\dots,9}$ the coefficient vector of the IO polynomial:

$$\ddot{y}^2 + \gamma_1 y^2 + \gamma_2 y\dot{y} + \gamma_3 y\ddot{y} + \gamma_4 y + \gamma_5 \dot{y}^2 + \gamma_6 \dot{y}\ddot{y} + \gamma_7 \dot{y} - \gamma_8 \ddot{y} + \gamma_9 = 0 \qquad (4)$$

- $\mathcal{C}$ the semi-algebraic set defined by $C(p)$ composed of all algebraic equations and inequalities verified by the components of the parameter vector $p = (p_1, \dots, p_5)$

to test

$$\left\{\begin{array}{lcl} p \in \mathcal{C} & & \\ \bar{p} \in \mathcal{C} & & \\ p_1 & = & \bar{p}_1, \\ p_3 & = & \bar{p}_3, \\ \phi(p) & = & \phi(\bar{p}) \end{array}\right. \Rightarrow \quad p_5 = \bar{p}_5.$$

$$\left\{\begin{array}{l} I = g m h (V - E), \\ \dfrac{dm}{dt} = \dfrac{m_\infty - m}{\tau_m}, \\ \dfrac{dh}{dt} = \dfrac{h_\infty - h}{\tau_h}, \end{array}\right. \iff \left\{\begin{array}{l} y = u\, p_5\, x_1\, x_2, \\ \dot{x}_1 = p_1\,(p_2 - x_1), \\ \dot{x}_2 = p_3\,(p_4 - x_2). \end{array}\right.$$

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- $\mathcal{C}$ the semi-algebraic set defined by $C(p)$ composed of all algebraic equations and inequalities verified by the components of the parameter vector $p = (p_1, \ldots, p_5)$

to test

$$S_{p_1,p_3} \cup \{p_5 \neq \bar{p}_5\} \quad \text{has no real solution.}$$

where
$S_{p_1,p_3} = C(p) \cup C(\bar{p}) \cup \{p_1 = \bar{p}_1, p_3 = \bar{p}_3\} \cup \{\gamma_k(p) = \gamma_k(\bar{p}) \mid k = 1, \ldots, 9\}$.

$\Longrightarrow$ development of a method and the algorithm *IdentifiabilityTree*.

$$\begin{cases} I = gmh(V-E), \\ \dfrac{dm}{dt} = \dfrac{m_\infty - m}{\tau_m}, \\ \dfrac{dh}{dt} = \dfrac{h_\infty - h}{\tau_h}, \end{cases} \quad \underset{\substack{x_1 = m,\, x_2 = h,\ u = V - E \\ p_1 = \frac{1}{\tau_m},\ p_2 = m_\infty, \\ p_3 = \frac{1}{\tau_h},\ p_4 = h_\infty,\ p_5 = g,}}{\Longleftrightarrow} \quad \begin{cases} y = u\,p_5\,x_1\,x_2, \\ \dot{x}_1 = p_1\,(p_2 - x_1), \\ \dot{x}_2 = p_3\,(p_4 - x_2). \end{cases}$$

Results of the IdentifiabilityTree algorithm:

- One of the branch: $[\not{p_2}, \not{p_4}, p_5, \not{p_3}, p_1]$
- Two groups of parameters $\{p_2, p_4, p_5\}$ and $\{p_1, p_3\}$.
- Determination of the two parameters p_2, p_4 and the parameter p_3 ensures the identifiability of all the parameters.

The voltage clamp experiment:

1. Estimate the triplet $\{p_2, p_4, p_5\}$ from $y = up_5x_1x_2$ in fixing the voltage at different values and measuring the transmembrane current trace (V constant: $I = (V-E)p_5\,p_2\,p_4$);
2. Estimate p_1 and p_3 at a particular voltage value dependence.

Conclusion

Identifiability study:

- ✓ Ensures good properties to the mathematical model;
- ✓ Extension of this definition;
- ✓ Other examples: strategy to reparametrize unidentifiable ODE models into identifiable ones (Evans 2000, Meshkat 2011)....

The models

$$\begin{cases} \dot{x}(t,p,f) = g(x(t,p),u(t),f,p), \\ y(t,p,f) = h(x(t,p),u(t),f,p), \\ x(t_0,p,f) = x_0, \\ t_0 \le t \le T. \end{cases} \quad (6)$$

Definitions

- ✓ **A fault** is an unpermitted deviation of at least one parameter of the system from the acceptable standard condition.
- ✓ **Fault diagnosability** establishes which faults can be discriminated using the available sensors in a system.
- ✓ **Fault diagnosis** consists in fault detection of the malfunction of a system and the fault isolation of the faulty component.

$f = 0$ means no fault. In the case of uncontrolled models $u = 0$.

Example: Mass ($m = 1$) attached to an elastic spring (force k) u external force ($\not\equiv 0$), $d \geq 1$

$$\ddot{y} + k(f_1 - 1)^2\, y - (d + f_2)u = 0 \qquad \phi(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$$

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Algebraic signature

$ASig(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$, in particular:

- ✗ $ASig(f_{\{1\}}) = (k(f_1 - 1)^2, -d)$ et $ASig(f_{\{1,2\}}) = (k(f_1 - 1)^2, -d - f_2)$.
- ✗ $ASig(f_{\{1\}}) \cap ASig(f_{\{1,2\}}) = \emptyset$ for all $f_2 \in (0, 2)$, $-d \neq -d - f_2$.

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$$\ddot{y} + k(f_1 - 1)^2 y - (d + f_2)u = 0 \qquad \phi(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$$

Algebraic signature

$ASig(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$, in particular:

- ✗ $ASig(f_{\{1\}}) = (k(f_1 - 1)^2, -d)$ et $ASig(f_{\{1,2\}}) = (k(f_1 - 1)^2, -d - f_2)$.
- ✗ $ASig(f_{\{1\}}) \cap ASig(f_{\{1,2\}}) = \emptyset$ for all $f_2 \in (0, 2)$, $-d \neq -d - f_2$.

Definitions

- ✗ Two sets of faults are said **algebraic discriminable** if there exists an algebraic signature, such that, for all input u, the two signatures have an empty intersection.
- ✗ If all the distinct sets of faults are algebraic discriminable, the model is said **algebraically diagnosable**.

Example: Mass ($m = 1$) attached to an elastic spring (force k) u external force ($\not\equiv 0$), $d \geq 1$

$$\ddot{y} + k(f_1 - 1)^2\, y - (d + f_2)u = 0 \qquad \phi(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$$

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Remark

The current algebraic signature is not sufficiently discriminant!

$$P(y,u,p,f) = m_0(y,u) + \sum_{k=1}^{q} \gamma_k(p,f) m_k(y,u) = 0 \quad \text{and} \quad \begin{cases} \gamma_1(p,f) = \phi_1, \\ \vdots \\ \gamma_q(p,f) = \phi_q, \end{cases}$$

Algorithm *Algebraic-Signature*

- ✓ Groebner basis computation;

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- ✓ Groebner basis computation;

Example: $\ddot{x} + k(f_1 - 1)^2 x - (d + f_2)u = 0$, $f_1 \in [0,2)$, $f_2 \in [0,2)$
with $\phi(f) = (\phi_1, \phi_2) = (k(f_1 - 1)^2, -d - f_2)$.

Algorithm *Algebraic_signature*: $ASig(f) = (\phi_1 - k, \phi_2 + d)$.

Remarks

- ✓ Algebraic signature: each of its component depends only on ϕ_k and the parameters of the system;
- ✓ By construction, one of the component of the algebraic signature vanishes when at least one specific (multiple) fault occurs.

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How to certify the values of *ASig*?

$$ASig: \begin{array}{ccc} \mathbb{R}^e & \longrightarrow & (R[\phi_1, \ldots, \phi_N])^l \\ f & \mapsto & (ASig_1(\phi), \ \ldots, \ ASig_l(\phi)) \,. \end{array}$$

Notations: ($\mathcal{N}$ subset of $\{1, \ \ldots, \ e\}$)

- $C_{p,f}$ = set of all algebraic equations and inequalities verified by p and f.

  Example: $\ddot{x} + k(f_1 - 1)^2 x - (d + f_2)u = 0$, $ASig(f) = (\phi_1 - k, \phi_2 + d)$.

  $C_{p,f} = \{0 < k < 4, \ 1 \leq d, \ 0 \leq f_1 < 2, \ 0 \leq f_2 < 2\}$.

- $S_{\mathcal{N}} = \{\gamma_1(p, f) = \phi_1, \ \ldots, \ \gamma_q(p, f) = \phi_N\} \cup C_{p,f} \cup \{f_i \neq 0 | i \in \mathcal{N}\} \cup \{f_i = 0 | i \notin \mathcal{N}\}$.

  Example: $\ddot{x} + k(f_1 - 1)^2 x - (d + f_2)u = 0$, $ASig(f) = (\phi_1 - k, \phi_2 + d)$.

  If $\mathcal{N} = \{1\}$ then $S_{\mathcal{N}} = \{k(f_1 - 1) = \phi_1, (d + f_2) = \phi_2\} \cup C_{p,f} \cup \{f_1 \neq 0\} \cup \{f_2 = 0\}$;

  If $\mathcal{N} = \{1, 2\}$ then $S_{\mathcal{N}} = \{k(f_1 - 1) = \phi_1, (d + f_2) = \phi_2\} \cup C_{p,f} \cup \{f_1 \neq 0\} \cup \{f_2 \neq 0\}$.

$$\begin{array}{rccc} ASig: & \mathbb{R}^e & \longrightarrow & (R[\phi_1,\ldots,\phi_N])^l \\ & f & \mapsto & (ASig_1(\phi),\ \ldots,\ ASig_l(\phi))\,. \end{array}$$

Criterion

Two criterion to discriminate multiple fault signatures:

- For the multiple fault $f_{\mathcal{N}}$, $S_{\mathcal{N}} \cup \{ASig_k(f_{\mathcal{N}}) = 0\} = \emptyset$?
- When $ASig_j(f) = 0$ is equivalent to $f_i = 0$?

$\hookrightarrow$ Algorithm *ExpectedValuesOfASign*.

Example: $\ddot{x} + k(f_1 - 1)^2 x - (d + f_2)u = 0$, $ASig(f) = (\phi_1 - k, \phi_2 + d)$.

$$C_{p,f} = \{0 < k < 4,\ 1 \le d,\ 0 \le f_1 < 2,\ 0 \le f_2 < 2\}$$

f	$ASig_1(f)$	$ASig_2(f)$
$f_{\{\}}$	0	0
$f_{\{1\}}$	$\emptyset$	0
$f_{\{2\}}$	0	$\emptyset$
$f_{\{1,2\}}$	$\emptyset$	$\emptyset$

$$C_{p,f} = \emptyset$$

f	$ASig_1(f)$	$ASig_2(f)$
$f_{\{\}}$	0	0
$f_{\{1\}}$		0
$f_{\{2\}}$	0	$\emptyset$
$f_{\{1,2\}}$		$\emptyset$

Conclusion

Diagnosability study:

- ✓ from the data collected on the physical system, can the chosen mathematical model permit to discriminate predefined faults that may occur on the system?
- ✓ New example of the interest of computer algebra and semialgebraic approach;
- ✓ Precomputations lead to efficient numerical procedures $\rightarrow$ detect and isolate (multiple) faults.

Conclusion

Diagnosability study:

- ✓ from the data collected on the physical system, can the chosen mathematical model permit to discriminate predefined faults that may occur on the system?
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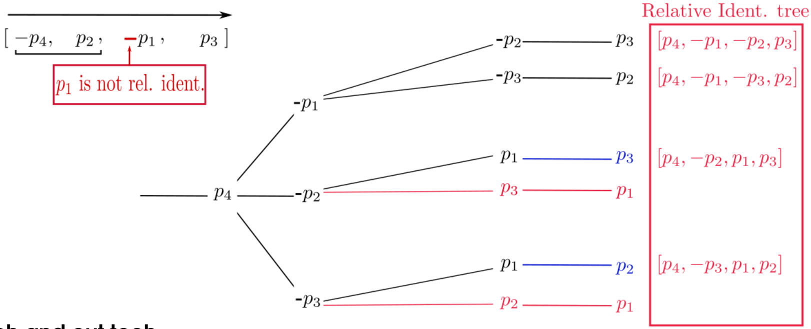
THANK YOU FOR YOUR ATTENTION and HAVE FUN WITH THE TUTORIAL!

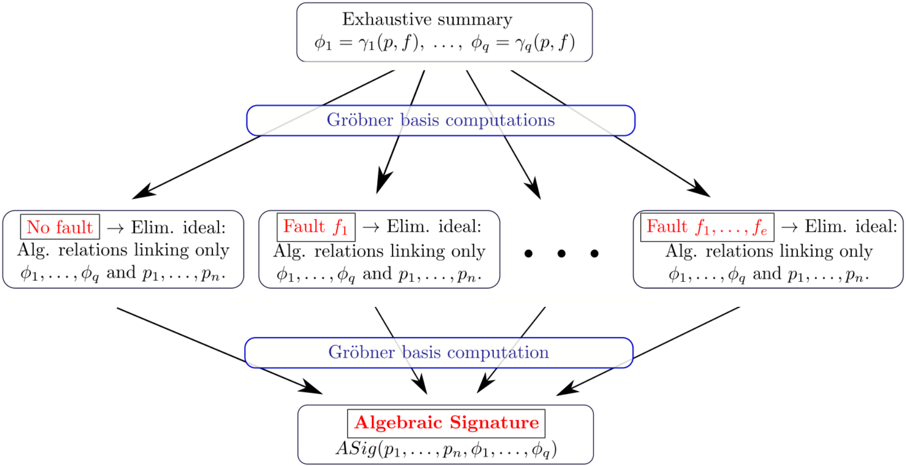
Procedure ***RelativeIdentifiabilityTree***
Objective: det. the keys param. which estimations turn the model into an ident. one.
Inputs: exhaustive summary, constraints on the parameters .
Output: A set of lists def. the relative identifiability tree $\mathcal{T}$ (any prefix *b* of $l \in \mathcal{T}$ is followed in *l* by an identifiable parameter wrt to *b* if there exists one.)
Convention: $-p$ in a list *l* means that *p* is not rel. identifiable wrt the param appearing before *p* in the list *l*.



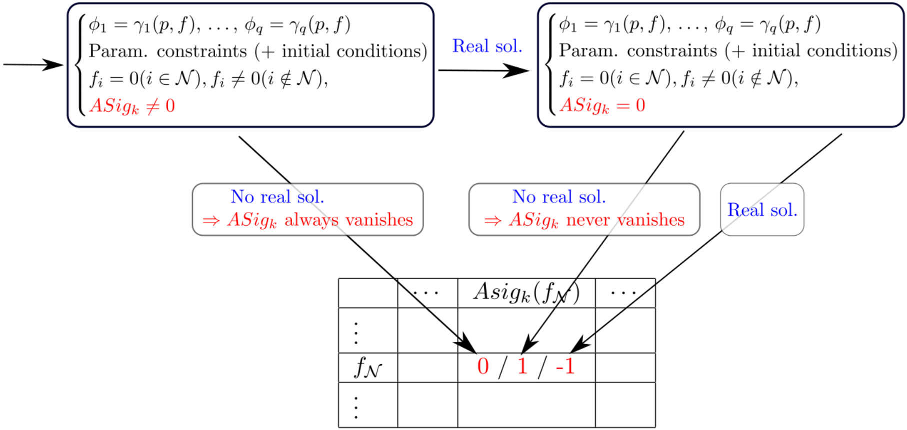
Branch and cut tech:

- Parameters relatively ident. wrt the same set of parameters can be permuted.
- The relative identifiability wrt a list of param. does not depend on the list of parameters but on the set they define.

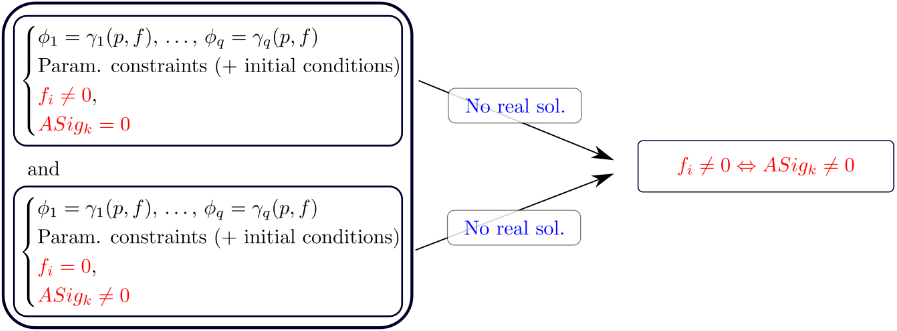
Procedure *ASign* - Computation of an algebraic signature
Objective: obtaining of polynomials discriminating the faults and depending only on the parameters and the components of the exhaustive summary.
Inputs: exhaustive summary, single faults list.
Output: an algebraic signature.



Procedure ***ExpectedValuesOfASign***
Objective: det. the (multiple) faults which can be discriminated with the alg. signature.
Inputs: Alg signature, exhaustive summary, single faults list, param constraints.
Outputs: the lists composed of a (multiple) faults and of the corresponding vector of expected values of the algebraic signature.



Procedure *SingleFaultCharacterization*
Objective: Reducing the number of tests needed to determine the table of the expect. values of the algebraic signature.
Inputs: Alg. signature, exhaustive summary, single faults list, parameters contraints.
Output: list of 2-uplets $[f_i, ASig_k]$ such that $f_i \neq 0 \Leftrightarrow ASig_k \neq 0$



$\rightarrow$ The output can be used as an optional argument in proc. *ExpectedValuesOfASign*.

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